1990 Mathcounts State Sprint and Target Rounds Solutions

SOLUTIONS TO SPRINT ROUND PROBLEMS

Problem 1. Solution: 64 days.

Since 1 pound is equivalent to 16 ounces, 40 pounds is equal to 40 × 16 ounces. There are 4 ounces in each cup of dog food, so Suzie's dog eats 4 × $2\frac{1}{2}$ ounces of dog food everyday. To find out how many days Suzie can feed her dog with a 40 pound bag, we divide the two: $\frac{40\times16}{2\frac{1}{2}\times4}=\frac{40\times16}{\frac{5}{2}\times4}=\frac{40\times16}{10}=4\times16=64$ days.

Problem 2. Solution: 30.

Let x be the number of thirty-sixths in $83\frac{1}{3}\%$.

$$x\times\frac{1}{36}=83\frac{1}{3}\% \qquad \Rightarrow \qquad x=36\times83\frac{1}{3}\%=36\times\frac{250}{300}=30.$$

There are 30 thirty-sixths in $83\frac{1}{3}\%$.

Problem 3. Solution: −12.5.

$$125\%\times10-(1.25\%\times1000+12.5\%\times100)=1250\%-1250\%-\frac{12.5}{100}\times100=-12.5.$$

Problem 4. Solution: −262.

Substituting in –3 for x, 5 for y, and –2 for z into the expression yields
$z-5(z(x-y)-12x)=-2-5[(-2)(-3-5)-12(-3)]=-2-5(16+36)=-262.$

Problem 5. Solution: $60a^2b^2$.

We want to find the LCM of the two expressions below.
$12a^2b$
$30ab^2$

$LCM(12, 30) = 60$
$LCM(a^2, a) = a^2$
$LCM(b, b^2) = b^2$
$LCM(12a^2b, 30ab^2) = 60a^2b^2$

(See "50 Mathcounts Lectures" Volume 2 Chapter 39 LCM/GCF)

Problem 6. Solution: $\frac{1}{221}$.

There are 4 kings in a standard deck of 52 cards..
The probability drawing a king in the first draw is $\frac{4}{52}$.
The probability drawing a king in the second draw is $\frac{3}{51}$.
The probability drawing two kings is $\frac{4}{52} \times \frac{3}{51} = \frac{1}{13 \times 17} = \frac{1}{221}$.

Problem 7. Solution: 80%.

Let W be the number of females and F be the number of people in favor of the issue.

The number of voters surveyed that were female or in favor of the issue is

$$n(W\ \ or\ \ F) = n(W) + n(F) - n(W \,\&\, F) = (348 + 469) + (348 + 482) - 348 = 1299.$$

According to the table, the total number of people surveyed is 348 + 469 + 482 + 326 =1625.

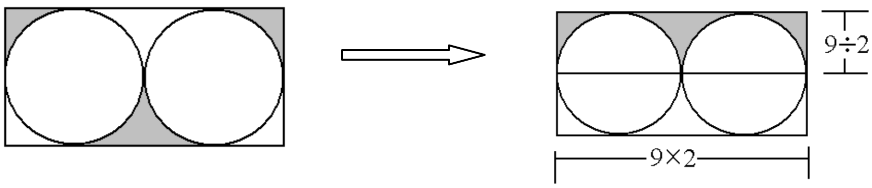
The percent of voters surveyed that were female or in favor of the issue is
$\frac{1299}{1625} \approx 0.799 \approx 80\%$.

Problem 8. Solution: −1.

$$\frac{z+3}{z+5}=\frac{z-2}{z-5}=\frac{(z+3)+(z-2)}{(z+5)+(z-5)}=\frac{2z+1}{2z}=\frac{(z+3)-(z-2)}{(z+5)-(z-5)}=\frac{5}{10}=\frac{1}{2}$$

$$\Rightarrow \quad \frac{2z+1}{2z}=\frac{1}{2} \quad \Rightarrow \quad \frac{2z+1}{z}=1 \quad \Rightarrow \quad 2z+1=z \quad \Rightarrow \quad z=-1 .$$

Problem 9. Solution: $81-\frac{81}{4}\pi$.



The area of the shaded region is $(9 \times 2)(9 \div 2) - \pi \times (9 \div 2)^2 = 81-\frac{81}{4}\pi$.

Problem 10. Solution: 18.

Since 7 is a solution, we have:
$7^2 - 4\times 7 = a \qquad \Rightarrow \qquad a = 7(7-4) = 7\times 3 = 21$.

According to Vieta's formulas, the sum of the solutions to the quadratic equation is negative the value of the coefficient of x^2 over the coefficient of x.
$b + 7 = -\frac{-4}{1} = 4$.
$\Rightarrow \quad b = -3$.
$a + b = 21 - 3 = 18$.

Problem 11. Solution: 36%.

Let the area of the original circle be A_1 and the area of the new circle be A_2. Let the radius of the original circle be r_1 and the radius of the new circle be r_2. Since the radius of the original circle is decreased by 20%, $r_2 = 0.8r_1$.

The area is decreased by $\frac{A_1 - A_2}{A_1}$%.

$$\frac{A_1 - A_2}{A_1} = 1 - \frac{A_2}{A_1} = 1 - \left(\frac{r_2}{r_1}\right)^2 = 1 - \left(\frac{0.8r_1}{r_1}\right)^2 = 1 - 0.8^2 = 1 - 0.64 = 0.36 = 36\%.$$

Problem 12. Solution: (5, 7.5).

For the area of triangle *ABD* to equal to the area of triangle *DBC*, *D* must be the midpoint of the line segment *AC*.

By the midpoint formula, we have

$x_D = \frac{x_A + x_C}{2} = \frac{10+0}{2} = 5$ and $y_D = \frac{y_A + y_C}{2} = \frac{15+0}{2} = 7.5$.

The coordinates of point *D* are (5, 7.5).

Problem 13. Solution: 270.

Let the weight of the ball 8 inches in diameter be W_1 and the volume be V_1.
Let the weight of the ball 12 inches in diameter be W_2 and the volume be V_2.

The weight of the ball is proportional to the volume of the ball.

$$\frac{W_2}{W_1} = \frac{V_2}{V_1} = \left(\frac{r_2}{r_1}\right)^3 = \left(\frac{12}{8}\right)^3 = \frac{27}{8} \qquad \Rightarrow \qquad W_2 = \frac{V_2}{V_1} \times W_1 = \frac{27}{8} \times 80 = 270 \text{ pounds.}$$

The iron ball 12 inches in diameter weighs 270 pounds.

Problem 14. Solution: $\frac{17}{450}$.

$0.03\overline{7} = \frac{37-3}{900} = \frac{34}{900} = \frac{17}{450}$.

(See "50 Mathcounts Lectures" Chapter 9. Converting among fractions, decimals, and percents).

Problem 15. Solution: –2.

Substituting in 0.5 for x into the equation $z = x^2 + 2$ yields
$z = x^2 + 2 = 0.5^2 + 2 = 2.25$.
$y = \frac{3z - z^3}{2} = \frac{z(3 - z^2)}{2}$.
$z^2 = 2.25^2 = 5.0625 \approx 5$.
$\frac{z(3 - z^2)}{2} = \frac{2.25(3 - 5)}{2} = -2.25 \approx -2$.
y to the nearest integer is –2.

Problem 16. Solution: 5 years.

Let x be the number of years it will be until Kathi and Jack earn the same hourly wage.

Since Kathi receives a \$.15 per hour pay raise every 6 months, in x years, Kathi will earn $\$3.55 + 0.15 \times 2 \times x$ per hour.

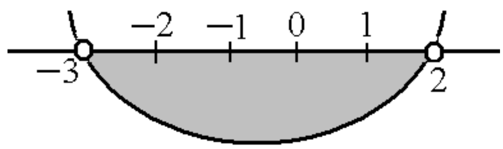
Since Jack has a \$.10 per hour pay decrease every 6 months, in x years, Jack will earn $\$6.05 - 0.1 \times 2 \times x$ per hour.

$3.55 + 0.15 \times 2 \times x = 6.05 - 0.1 \times 2 \times x \quad \Rightarrow \quad x = 5$.

It will take 5 years until Kathi and Jack earn the same hourly wage.

Problem 17. Solution: {-2, -1, 0, 1}.

The critical points of the equation are –3 and 2. Because (x + 3)(x – 2) must be less than 0, the solutions are bounded by the critical points, or in the gray region. The integer solutions to the equation are –2, –1, 0, and 1.



(See "Twenty more problem solving skills" Chapter 9 Absolute Values and Inequalities).

1990 Mathcounts State Sprint and Target Rounds Solutions

Problem 18. Solution: 203.

Look for a pattern:

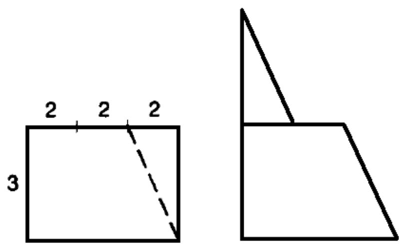
4	5	6	7	8	9	10	11	12	1	14
23	32	43	56	71	88	107	128	151	176	**203**

Differences: 9, 11, 13, 15, 17, 19, 21, 23, 25, 27

The output is 203.

Problem 19. Solution: 18 square units.

Both regions have the same area of 3 × (2 + 2 + 2) = 18.



Problem 20. Solution: $\frac{7}{11}$.

$$\frac{1}{2-\frac{1}{1+\frac{2}{1+\frac{1}{2}}}}=\frac{1}{2-\frac{1}{1+\frac{2}{\frac{3}{2}}}}=\frac{1}{2-\frac{1}{1+\frac{4}{3}}}=\frac{1}{2-\frac{1}{\frac{7}{3}}}=\frac{1}{2-\frac{3}{7}}=\frac{1}{\frac{14-3}{7}}=\frac{7}{11}.$$

Problem 21. Solution: $b=11\frac{1}{4}$.

Let b be the number of hours it will take for the slower typist to complete the paper alone.

$$\frac{1}{9}+\frac{1}{b}=\frac{1}{5} \quad\Rightarrow\quad \frac{1}{b}=\frac{1}{5}-\frac{1}{9}=\frac{9-5}{45}=\frac{4}{45}=\frac{1}{\frac{45}{4}}=\frac{1}{11\frac{1}{4}} \quad\Rightarrow\quad b=11\frac{1}{4}.$$

1990 Mathcounts State Sprint and Target Rounds Solutions

Problem 22. Solution: 88.

$11_2 + 11_3 + 11_4 + ... + 11_{12} = 1\times 2^1 + 1\times 2^0 + 1\times 3^1 + 1\times 3^0 + 1\times 4^1 + 1\times 4^0 + ... + 1\times 12^1 + 1\times 12^0$
$2+1+3+1+4+1+...+12+1 = (2+3+4+...+12) + (1\times 11) = 77 + 11 = 88.$

Problem 23. Solution: 10*a*.

Let the length of the line segment be *d*. By the distance formula,
$d = \sqrt{(9a-a)^2 + (7a-a)^2} = \sqrt{(8a)^2 + (6a)^2} = \sqrt{64a^2 + 36a^2} = \sqrt{100a^2} = 10a$

Problem 24. Solution: 676,000.

There are 26 choices for a letter and 10 choices for a number.
Since there are no restrictions to the number of occurrences of letters or numbers, by the fundamental counting principle, we have

26 × 10 × 10 × 10 × 10 × 26 = 676000 different codes.

Problem 25. Solution: 301.

The first term is 2, the common difference of the series is 5 – 2 = 3, and the number of terms is n = 14. The 14th term is $a_{14} = a_1 + (n-1))\times d = 2 + (14-1)\times 3 = 41$.
The sum of the first 14 terms is $S = \frac{(a_1 + a_{14})n}{2} = \frac{(2+41)\times 14}{2} = 301$.

Problem 26. Solution: 9.85.

Let the score that Mary needs receive an average score of 9.45 be *x*.

$$\frac{9.25 + 9.4 + 9.3 + 9.45 + x}{5} = 9.45$$

$\Rightarrow \quad 9.25 + 9.4 + 9.3 + 9.45 + x = 9.45 \times 5$

$\Rightarrow \quad 9.25 + 0.2 - 0.2 + 9.4 + 0.05 - 0.05 + 9.3 + 0.15 - 0.15 + 9.45 + x = 9.45 \times 5$

$\Rightarrow \; -0.2 - 0.05 - 0.15 + x = 9.45 \quad \Rightarrow \; -0.4 + x = 9.45 \quad \Rightarrow \; x = 9.45 + 0.4 = 9.85.$

1990 Mathcounts State Sprint and Target Rounds Solutions

Problem 27. Solution: 12 days.

Let the number of days it will take 60 workers to complete the same task be *x*.

$$\frac{1}{24\times10\times24}=\frac{1}{60\times8\times x} \quad\Rightarrow\quad 24\times10\times24=60\times8\times x \quad\Rightarrow\quad x=12.$$

(See "Twenty more problem solving skills for Mathcounts" Chapter 14. Words problems related to work).

Problem 28. Solution: $T-P\times X$.

The total number of points scored in the games the team won is $P\times X$.

Since the total number of points scored in the season is T, the total number of points scored in the games the team lost is $T-P\times X$.

Problem 29. Solution: 34.

Method 1:

$K\equiv4 \mod 5 \quad\Rightarrow\quad K\equiv34 \mod 5 \qquad (1)$

$K\equiv2 \mod 8 \quad\Rightarrow\quad K\equiv34 \mod 8 \qquad (2)$

$K\equiv1 \mod 11 \quad\Rightarrow\quad K\equiv34 \mod 11 \qquad (3)$

(1), (2), and (3) become: $K\equiv34 \mod \text{LCM}(5, 8, 11)$

or $K\equiv34 \mod 440$.

The smallest value for *K* is 34.

Method 2:

$K=5q+4$	$\Rightarrow$	K:	9	14	19	24	29	34	39...
$K=8r+2$	$\Rightarrow$	K:	10	18	26	34	42	50...	
$K=11s+1$	$\Rightarrow$	K:	12	23	34	45...			

The smallest common value for *K* is 34.

Problem 30. Solution: 18.

Let the semicircle have a diameter of *d.*

$$\frac{\frac{\pi d^2}{4}}{2}=\frac{49\pi}{8} \qquad \Rightarrow \qquad d=7.$$

The perimeter of the semicircle to the nearest whole number is

$$d+\frac{\pi d}{2}=7+\frac{7\pi}{2}\approx 17.99\approx 18.$$

1990 Mathcounts State Sprint and Target Rounds Solutions

SOLUTIONS TO TARGET ROUND PROBLEMS

Problem 1. Solution: 8.

Let x be the number of dollars per pizza and n be the number of pizzas R. B. bought.

$xn = 56$

$(x-1)(n+1) = 56 \quad \Rightarrow \quad xn - n + x - 1 = 56$

$\Rightarrow \quad 56 - n + x - 1 = 56 \quad \Rightarrow n + 1 = x \quad \Rightarrow (x-1)x = 56 \quad \Rightarrow x = 8$

R.B. paid 8 dollars per pizza.

Problem 2. Solution: 420.

The smallest possible integer divisible by 2, 3, 7, 10, 15, 20, 21, and 28 is the same as the least common multiple of the numbers.

$2 = 2$

$3 = 3$

$7 = 7$

$10 = 2 \times 5$

$15 = 3 \times 5$

$20 = 2^2 \times 5$

$21 = 3 \times 7$

$28 = 2^2 \times 7$

The least common multiple is $2^2 \times 7 \times 5 \times 3 = 420$.

Problem 3. Solution: 1/17.

$$\left(\left(\frac{1}{2}\right)^{-1} + \left(\frac{1}{3}\right)^{-1} + \left(\frac{1}{5}\right)^{-1} + \left(\frac{1}{7}\right)^{-1}\right)^{-1} = (2+3+5+7)^{-1} = \frac{1}{17}.$$

Problem 4. Solution: 3.

$|3\#9| = |2 \times 3 - 9| = 3$

$6@3=\frac{6+3}{3}=3$.

Problem 5. Solution: 10.

$8575=5^2\times 7^3$
8575 has (2 + 1)(3 + 1)= 12 factors. Among these factors, 2 are prime numbers (5 and 7).

The number of factors of 8575 that are not prime is 12 – 2 = 10.

Problem 6. Solution: 1200.

Let the number of packages made by ten robots in ten hours be *x*.

$$\frac{\text{Number of packages}}{\text{number of robots}\times\text{number of minutes}}=\frac{5}{5\times 5}=\frac{x}{10\times 10\times 60} \quad\Rightarrow\quad x=1200.$$

Problem 7. Solution: 217.

Method 1 (Indirect way):
There are 9 digits available: 1, 2, 3, 4, 5, 6, 7, 8, and 9.
There are 9 × 9 × 9 = 729 3-digit numbers using no zeros.
There are 8 × 8 × 8 = 512 3-digit numbers using no zeros and no 7's.
The number of 3-digit numbers that contain no zeros and at least one 7 is 729 – 512 = 217.

Method 2 (Direct way):
Case I: One 7 is used: There are $1\times 8\times 8\times\frac{3!}{1!\times 2!}=192$ 3-digit numbers.
Case II: Two 7's are used: There are $1\times 1\times 8\times\frac{3!}{1!\times 2!}=24$ 3-digit numbers.
Case III: Three 7's are used: There is only one such 3-digit number (777).
There are 192 + 24 + 1 = 217 3-digit numbers that contain no zeros and at least one 7.

Problem 8. Solution: $\frac{1}{2}$.

The shortest altitude, h_S, is perpendicular to the side of length 80 and the longest altitude, h_L, is perpendicular to the side of length 40.

$$\frac{1}{2}b_1h_1 = \frac{1}{2}b_2h_2 \quad \Rightarrow \quad \frac{1}{2}\times 40h_L = \frac{1}{2}\times 80h_S \quad \Rightarrow \quad \frac{h_S}{h_L} = \frac{40}{80} = \frac{1}{2}.$$

The shortest altitude is $\frac{1}{2}$ times the longest altitude.