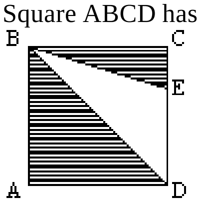
2000 CHAPTER COMPETITION

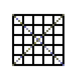
SPRINT ROUND

1. $\frac{220}{55} = 4$ **Answer**
2. If there are 36 stairs between the 1st and 3rd floors (i.e., two floors), then there must be 18 stairs between the 1st and 2nd floors and 18 stairs between the 2nd and 3rd floors. There are 5 floors between the 1st and 6th floor. $5 \times 18 = 90$ **Answer**
3. $\sqrt{(8)(32)} = \sqrt{(2^3)(2^5)} = \sqrt{2^8} = 2^4 =$ 16 **Answer**
4. There are 16 boys and 24 girls in the class for a total of 16 + 24 = 40 students. $\frac{24}{40} = \frac{6}{10} = 60\%$ **Answer**
5. If you number the points on the circumference A, B, ..., F, G, then the chords would be named AB, AC, etc., BA, BC, etc. Remember, though, that half of the chords will be exactly the same, just with the names of the points reversed, e.g., BA (for AB), CA (for AC), etc. For n points on a circumference, there will be $\frac{n \times (n-1)}{2}$ chords. In this case, there are 7 possibilities for the start of the chord and 6 possibilities for the end of the chord or $7 \times 6 = 42$ chords. However, taking into account each chord has been enumerated twice: $\frac{42}{2} = 21$ **Answer**
6. Square ABCD has an area of 36.



Triangle BCD has half of the area of the square. Triangle BCE is $\frac{1}{3}$ the area of triangle BCD (since the area of a triangle is $\frac{1}{2}bh$, BC is the height and DE = 2EC). Therefore, the area of triangle BED is $\frac{2}{3}$ of the area of triangle BCD and the ratio of triangle BED to the area of the square = $\frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$ **Answer**

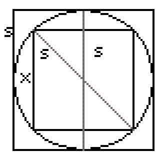
7. Rocky must move a total of two squares forward and two squares backward. This means he must get two heads and two tails. Let's list the combinations:
HHTT, HTHT, THHT, THTH, TTHH, HTTH
There are $2^4 = 16$ total combinations.
$\frac{6}{16} = \frac{3}{8}$ **Answer**
8. Since we have square tiles filling up a square, we're talking about $n \times n$ or n^2 tiles. The first diagonal will go through the first tile in the first row, the second tile in the second row and so on. The second diagonal will go through the n^{th} diagonal in the first row all the way to the first tile in the n^{th} row. If n is odd, the diagonals will go through the same tile and that tile will be the middle tile in the middle row. If n is even, the diagonals will not go through the same tile and the number of tiles touched by the diagonal will be even. But we're told the total number of tiles is 9, which is odd.
$2n - 1 = 9$
$2n = 10$
n = 5 so the total number of tiles on the floor are $n^2 = 5 \times 5 = 25$ **Answer**
Here's the picture:



9. We have 3 blue, 4 red and 3 yellow marbles. We need to construct a fraction containing the blue marbles in the numerator, and the total number of marbles in the denominator that equates to $\frac{3}{4}$, which is the equivalent of 75%. Let x = the number of blue marbles that are needed for this to happen.
$\frac{3+x}{10+x} = \frac{3}{4}$
$12 + 4x = 30 + 3x$
$x = 18$ **Answer**
10. There are $4! = 4 \times 3 \times 2 \times 1 = 24$ possible combinations. But how do we know which ones are a multiple of 4? Since 100 is a multiple of 4, any number ending in 00 is a multiple of 4. And we can add 4 to that number and get another multiple of 4 (e.g., 104 is a multiple of 4 as is 204, 304 etc.) The same is true for 08, 12, 16, etc.

Therefore, any number whose last two digits (tens and digits column) are a multiple of 4 is a multiple of 4. So, let's list the set of two digits that are multiples of 4: 12, 24, 32 (40 and 44 can't work here). In each case, there are 2 different values for each choice (e.g., 3412, 4312). $\frac{2 \times 3}{24} = \frac{6}{24} = \frac{1}{4}$ **Answer**

11. Let s = the side of the larger square.

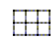


The area of the larger square is 6.
Let x = the size of the smaller square.
s is also the length of a parallel line to the side going through the center of the circle. This means that s is the diameter of the circle and, therefore, s is also the diagonal of the smaller square and the hypotenuse of a 45-45 right triangle.
$x^2 + x^2 = s^2$
$2x^2 = s^2 = 6$
$x^2 = 3$ **Answer**

12. Anthony can cut a lawn in 2 hours so he does $\frac{1}{2}$ a lawn per hour. Mia can cut the lawn in 3 hours so she does $\frac{1}{3}$ of the lawn per hour. Dandria cuts the lawn at the same rate as Anthony. Anthony starts by cutting the lawn for $\frac{1}{2}$ hour so he gets $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ of the lawn done. Mia cuts the lawn for an hour and so completes another $\frac{1}{3}$ of the lawn. This means that $\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$ of the lawn has been cut. This leaves $1 - \frac{7}{12} = \frac{5}{12}$ of the lawn to go. Mia can do $\frac{1}{2} = \frac{6}{12}$ of the lawn in 1 hour, or 60 minutes.
$\frac{\frac{5}{12}}{\frac{6}{12}} \times 60 = \frac{5}{6} \times 60 = 50$ **Answer**

13. The two superstars must play leaving us 10 players from which to choose 3 more. There are $10 \times 9 \times 8 = 720$ different ways to choose 3 people. However, for each choice, there are $3 \times 2 \times 1 = 6$ ways to represent them that are equivalent. Therefore, there are $\frac{720}{6} = 120$ line-ups. **Answer**

14. First, count up all the individual squares created by connecting each dot to its neighbors (i.e., the side of the square is the length of one dot to the next). There are 6.

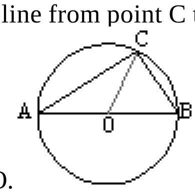


Now, count up all the squares created from making the sides of the squares the twice the length of the sides of the smaller squares. There are 2.

We're not done yet. There are 2 more! You have to look at the squares that can be obtained by creating diagonals.

6 + 2 + 2 = 10 **Answer**

15. Draw a line from point C to the center of the circle, O.



Since the radius of the circle is 5, OC = OB = BC = 5 and we have an equilateral triangle. This means that all angles are 60°. Since the number of degrees in a semicircle is 180°, arc BC must be $\frac{1}{3}$ of the circumference of the semicircle.
$\frac{1}{3}\Pi r = \frac{1}{3}\Pi 5 = \frac{5\Pi}{3}$ **Answer**

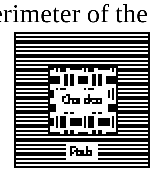
16. If the Rangers are to have twice as many wins as losses (and as a Pittsburgh Penguins fan, I don't like even thinking about it -- go Pens!) they must win $\frac{2}{3}$ of all 36 games, for a total of 24 games. Let x = the number of games they still need to win.
$5 + x = 24$
$x = 19$ **Answer**

17. Phil gave 20% of his winnings to his parents. This means he had 80% left. He gave 25%, or $\frac{1}{4}$ of the remaining money to his kids. $\frac{1}{4} \times 80\% = 20\%$ so he has 80% - 20% = 60% left.

$\frac{3}{5}x = 900000$
3x = 4,500,000
x = 1,500,000 **Answer**

18. A triangle has sides of lengths 4, 6, and 9. A similar triangle has one side with length 36. When triangles are similar, their lengths are proportional. To maximize the perimeter of this similar triangle, let the side of length 36 match the size of length 4. Then the similar triangle will have sides $\frac{36}{4} = 9$ times larger than the first triangle (versus 6 or 4 times larger, if you used the other sides).
4 + 6 + 9 = 19
$19 \times 9 = 171$ **Answer**

19. Let x be the cost of the ticket. 20 students bought tickets at a cost of 20x. The 10 additional students bought tickets at a cost of 10x. This money was received as a refund of $3.00 per student.
20 + 10 = 30 students
$30 \times 3 = 90$
$90 is the value of the refund and also the cost of 10 tickets.
10x = 90
x = 9
The cost of each ticket is $9.
$9 \times 20 = 180$ **Answer**

20. The perimeter of the square garden is 64 meters.  Therefore, the side of the garden is $\frac{64}{4} = 16$ meters. The area of the path is 228 square meters. The entire figure, i.e., garden and path together form a square. The area of this larger square is $16^2 + 228 = 256 + 228 = 484$. The side of this larger square is $\sqrt{484} = 22$. The amount of fencing needed to surround the outer edge of the path is just the perimeter of the larger square. $22 \times 4 = 88$ **Answer**

21. We have to choose one integer from the set {1, 2, 3, 4}and one integer from the set {5, 6, 7, 8}. There are $4 \times 4 = 16$ combinations and the largest sum we can get is 4 + 8 = 12. Which combinations sum to 8? (1,7), (2,6), (3,5) for a total of 3. Which sum to 9? (1,8), (2,7), (3,6), (4,5) for a total of 4. Which sum to 10? (2,8), (3,7), (4,6) for a total of 3. Which sum to 11? (3,8), (4,7) for a total of 2. Which sum to 12? (4,8) for a total of 1.
1 + 2 + 3 + 4 + 3 = 13
$\frac{13}{16}$ **Answer**

22. $6! = 2^4 \bullet 3^2 \bullet n$
$6 \times 5 \times 4 \times 3 \times 2 \times 1 =$
$2 \times 2 \times 2 \times 2 \times 3 \times 3 \times n =$
$(2 \times 3) \times (2 \times 2) \times 3 \times 2 \times 1 \times n =$
$6 \times 4 \times 3 \times 2 \times 1 \times n$
n = 5 **Answer**

23. The average of 12 different positive integers is 12. That means the sum of the integers must be $12 \times 12 = 144$. If one of the values is to be the largest possible, then the others must be the smallest possible.
1 + 2 + ... + 10 + 11 = 66
144 - 66 = 78 **Answer**

24. A cyclist takes 60 + 24 = 84 minutes to ride a course at 9 miles per hour or $\frac{9}{60} = \frac{3}{20} = 0.15$ miles per minute. Thus the entire course must be $0.15 \times 84 = 12.6$ miles. The second cyclist averages 6 miles per hour or $\frac{6}{60} = \frac{1}{10} = 0.1$ miles per minute. Therefore, he must take $\frac{12.6}{0.1} = 126$ minutes to travel the course.
$\frac{126}{60} = 2.1$ **Answer**

25. The overall average was 80 for a total aggregate score of $80 \times 30 = 2400$ points. Let x = the number of passing students.
84x + 60(30 - x) = 2400
84x + 1800 - 60x = 2400
24x = 600
x = 25 **Answer**

26. First, find the third side of the triangle.
$9^2 + x^2 = 41^2$
$81 + x^2 = 1681$
$x^2 = 1600$
x = 40
$A = \frac{1}{2} \times 40 \times 9 = 180$ **Answer**

27. Let's enumerate the two-digit numbers starting with 24 whose digital sum is 7: 25, 34, 43, 52, 61, and 70. That's 6 numbers. Now look at the two-digit numbers whose digital sum is 14: 59, 68, 77, 86, 95, or 5 numbers. There can't be any two digit numbers whose digital sum is 21 since you can't go higher than 9 + 9 = 18. So, now let's look at the three digit numbers, through

125, whose digital sum is 7: 106, 115, 124. That's 3 numbers. Now look for those whose digital sum is 14. There aren't any. Why? Because there can only be "1" hundred and no more than "2" tens. 1 + 2 = 3. We'd have to have some single digit value to represent 11, but there is none in base 10. So, 6 + 5 + 3 = 14 **Answer**

28. There are 120 students applying for the award. 10 have never studied either French or Spanish. That means that 110 students have studied either French, Spanish or both. 81 + 95 = 176 students that have studied French and/or Spanish. But there are only 110 students which means that 176 - 110 = 66 students must be studying both of them. 66 **Answer**

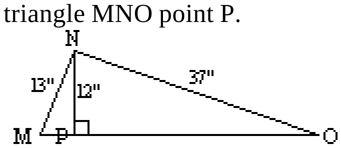
29. $11 + 12 + 13 + ... + 39 + 40 =$
$(11 + 40) + (12 + 39) + ... + (25 + 26) =$
$51 \times 15 = 765$ **Answer**

30. If the three lengths are a, b, and c, the volume will be $a \times b \times c$.
$54 \times 72 \times 108 = a \times b \times b \times c \times c \times a$
$(2 \times 3^3) \times (2^3 \times 3^2) \times (2^2 \times 3^3) =$
$2^6 \times 3^8 = a^2b^2c^2 = (abc)^2$
$abc = 2^3 \times 3^4 = 8 \times 81 = 648$ **Answer**

TARGET ROUND

1. Let x be the number Michelle chose.
$2x + x = 9639$
$3x = 9639$
$x = 3213$ **Answer**

2. Let's call the other end of the height of triangle MNO point P.



Then triangle MNP is a right triangle and MP = 5. Triangle PNO is a right triangle. We need to determine the length of PO.
Let x = PO.
$x^2 + 12^2 = 37^2$
$x^2 + 144 = 1369$
$x^2 = 1225$
$x = 35$
$MO = 5 + 35 = 40$
$A = \frac{1}{2} \times 12 \times 40 = 6 \times 40 = 240$ **Answer**

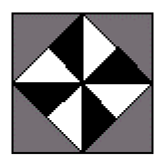
3. $f(x) = 3x + 7$
$g(x) = x - 1$
$g(2) = 2 - 1 = 1$
$f(1) = (3 \times 1) + 7 = 3 + 7 = 10$ **Answer**

4. Let's list all positive two-digit integers for which the sum of the digits is a multiple of 5. Start first with adding up to 5:
14, 23, 32, 41, 50
None of these gives a remainder of 5 when divided by 7. So try adding up to 10:
19, 28, 37, 46, 55, 64, 73, 82, 91
19 works ($19 = 2 \times 7 + 5$) and so does 82 ($82 = 11 \times 7 + 5$). No try adding up to 15:
69, 78, 87, 96
96 works ($96 = 13 \times 7 + 5$).
19 + 82 + 96 = 197 **Answer**

5. Let Q represent the quarters, D, the dimes, N, the nickels and P, the pennies.
$Q > D > N > P$
$Q + D + N + P = 10$
Since the number of coins for each type of coin must differ, if you start with P = 1, N = 2, D = 3 and Q = 4, you get a total of 10 coins.
$\frac{4}{10} = \frac{2}{5}$ **Answer**

6. The sequence is -2, 7, 9, 2, -7, -9, -2, etc. so the sequence repeats every 6 digits.
-2 + 7 + 9 + 2 + -7 + -9 = 0
$6 \times 6 = 36$; the first 36 elements in the sequence will sum to 0. Therefore, we need only add up the 37th through 40th elements.
-2 + 7 + 9 + 2 = 16 **Answer**

7. The side length of the largest square below is 10 inches.



Thus, the area is $10 \times 10 = 100$. The smaller square is broken up into 8 triangles all of whom have the same area, so that the shaded area inside the smaller square is half the area of the smaller square. Each of the four triangles shaded in gray, has sides of 5, so the total area in gray is $\frac{1}{2} \times 5 \times 5 \times 4 =$
$2 \times 5 \times 5 = 50$
Therefore, the smaller square must have area 100 - 50 = 50 and half of that is 25. **Answer**

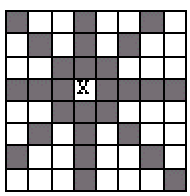
8. If the mean of the 3 values is greater than or equal to 3, the sum of the 3 values must be at least $3 \times 3 = 9$. There are a total of $5 \times 4 \times 3 = 60$ combinations. But each combination (a, b, c) can be expressed in 6 different ways, so there are only 10 ways that are meaningful. Let's list the ones that

will sum up to at least 9. Start with 9: (1, 3, 5), (2, 3, 4) That's 2. Now 10: (1, 4, 5), (2, 3, 5) That's 2 more. Now 11: (2, 4, 5) That's 1 more. And finally 12: (3, 4, 5) And 1 more. There are a total of 2 + 2 + 1 + 1 = 6 combinations that work.

$\frac{6}{10} = \frac{3}{5}$ **Answer**

TEAM ROUND

1. The queen can move to the 3 squares above and the 4 below for a total of 7. She can move to the 3 squares to the left and the 4 squares to the right for a total of 7 more. She can move diagonally left and up to 3 squares and diagonally right and down to 4 squares for a total of 7 more. Finally, she can move diagonally right and up to 3 squares and diagonally left and down for 3 squares for a total of 6 squares. See the gray shaded boxes in the figure below.



7 + 7 + 7 + 6 = 27 **Answer**

2. The probability of waiting n minutes where $1 \le n \le 20$ is $\frac{1}{20}$. There are 12 values of n (those < 13) that we don't have to deal with.
20 - 12 = 8
$\frac{8}{20} = \frac{2}{5}$ **Answer**

3. The Yellow cab costs $15 \times 4 = 60$ cents per hour after the first half-mile. The Blue Top cab costs $25 \times 2 = 50$ cents per hour after the first half-mile. Let x = the number of additional miles (after the first half-mile).
50 + 60x = 95 + 50x
10x = 45
x = 4.5 miles.
However, we have to add in the initial half-mile. 4.5 + 0.5 = 5 **Answer**

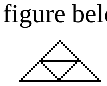
4. Let's just list them: first, take them separately: 1, 5, 10, 25 -- that's 4. Next, take sums of 2 coins: 6, 11, 26, 15, 30, 35 -- that's 6 more. Now take sums of 3 coins: 16, 31, 36, 40 -- or 4 more. Finally, add them all up: 41 or 1 more.
4 + 6 + 4 + 1 = 15 sums **Answer**

5. We remove every other card starting with the first and the removed cards are in order from 1 to 9. When we've gone through the first ten cards, 1, 2, 3, 4, and 5 are gone which means that the original form was:
1, a, 2, b, 3, c, 4, d, 5, e
Now we have 5 cards left and the order is:
a, b, c, d, e
This time a, c, and e will be removed leaving b and d. Therefore the original order was:
1, 6, 2, b, 3, 7, 4, d, 5, 8
The next card is b, but we just removed e and we must leave a card. Therefore, we will remove d, making d = 9 and b = 10.
Therefore, the original order was:
1, 6, 2, 10, 3, 7, 4, 9, 5, 8
The cards adjacent to 10 are 2 and 3.
2 + 3 = 5 **Answer**
Note: it's easy to see if you write them out:
1 x 2 x 3 x 4 x 5 x
6 x 7 x 8
x 9
10

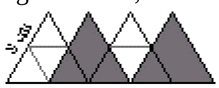
6. For a specific integer value of x,
$$2x + 9 = \frac{1}{x+5}$$
The expression $\frac{1}{x+5}$ will be a fraction for any integral value of x except -4 (and -5, but 1 divided by 0 is indeterminate). In this case the expression will be 1. The expression 2x+9 will also be integral for any integral value of x. Therefore,
2x + 9 = 1
Again, x = -4 **Answer**

7. One cyclist travels 4 miles faster than the other. If x is the speed that the slower one travels, x+4 is the speed that the faster one travels and in one hour they will be x + 4 + x = 2x + 4 miles closer to each other.
They meet after 45 minutes or $\frac{3}{4}$ of an hour.
$$(2x + 4)\frac{3}{4} = 13.5$$
$(2x + 4) \times 3 = 54$
6x + 12 = 54
6x = 42
x = 7 **Answer**

8. An equilateral triangle can be broken up into four smaller equilateral triangles as in the figure below.



In order for this to happen, the middle triangle must bisect all three sides of the larger triangle. In the figure in the problem, we have a similar case. Since the triangles intersect at the midpoint of the base, we end up forming smaller equilateral triangles whose sides are half the sides of the larger triangles. We need to determine just how many large triangles we really do have. We can do this by breaking up the large triangles in the picture into smaller ones. In the figure below, we have colored in two large



triangles in gray, and shown the rest of the area in terms of the smaller equilateral tirangles. There are 3 in the first one and 2 in the third for a total of 5 smaller triangles. It takes 4 smaller triangles to make up a larger one so we really have $3\frac{1}{4}$ large triangles. Now, we need to determine the area of one of the triangles.

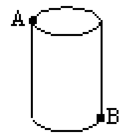
$A = \frac{1}{2} \times 8 \times \frac{8\sqrt{3}}{2} = 4 \times 4\sqrt{3} = 16\sqrt{3}$

$16\sqrt{3} \times 3\frac{1}{4} = 16\sqrt{3} \times \frac{13}{4} = 4\sqrt{3} \times 13 =$

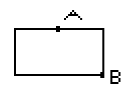
$52\sqrt{3}$ **Answer**

9. The easiest way to solve this is to determine how many numbers between 1 and 1,000,000 are perfect squares and perfect cubes. Any positive integer ≤ 1,000,000 whose square root is an integer is a perfect square that we can count. 1000^2 = 1,000,000 which means that there are 1000 perfect squares (1001^2 is too large). Any positive integer ≤ 1,000,000 whose cube root is an integer is a perfect cube that we can count. 100^3 = 1,000,000 which means that there are 100 perfect cubes. However, it's obvious that we have some overlap. (Just look at 1,000,000; we counted it as both a perfect square and a perfect cube.) We can determine the overlap by looking at the number, 1,000,000.
$1{,}000{,}000 = 10^6 = (10^2)^3 = (10^3)^2$
A perfect sixth will always be both a perfect cube and a perfect square. We need to determine how many integers exist whose sixth power is no more than 1,000,000.
$10^6 = 1{,}000{,}000$
Therefore, there are 10 integers which, when taken to the sixth power yield values in the correct range. 1000 + 100 - 10 = 1090
1000000 - 1090 = 998910 **Answer**

10. The ant needs to crawl from point A to point B.



The diameter of the cylinder is $3\frac{3}{4}$ which makes the circumference $\frac{15}{4}\Pi$. To figure out the distance from A to B, we take the cylinder and look at it like, perhaps, the cardboard cylinder inside a roll of paper towels. The cylinder can be cut open vertically to produce a rectangle. If we cut the cylinder at point B the rectangle looks as follows:



A is halfway around the circumference from B, therefore A is the midpoint on the top vertical line of the rectangle. Thus, we have a $8 \times \frac{15}{8}\Pi$ rectangle and we need to find the hypotenuse. Let x = the hypotenuse.

$8^2 + \left(\frac{15}{8}\Pi\right)^2 = x^2$

$64 + (5.892...)^2 = 64 + 98.7257... = x^2$

x = 9.93... ≈ 9.9 **Answer**