2000 STATE COMPETITION

SPRINT ROUND QUESTIONS

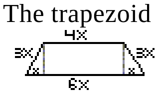
1. There are 2 E's and 2 A's in the word AVERAGE. If there were no duplicates, then there would be 7! possibilities. If there was only one duplicate letter, then there would be just half that number. (Suppose A was the duplicate and call the two A's, A1 and A2. Then, we would have a duplicate for each word, e.g., A_1VERA_2GE and A_2VERA_1GE.) But we have an additional duplicate letter, so half of the half, or one quarter of the 7! possibilities are valid.
$\frac{7!}{2 \times 2} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4} =$
$7 \times 6 \times 5 \times 3 \times 2 \times 1 = 7 \times 180 =$
1260 **Answer**

2. Joey runs 2 miles and walks 1 mile. Then he repeats this. He runs a mile in 8 minutes and walks a mile in 16 minutes. Therefore, to go 3 miles, he takes $2 \times 8 + 16 = 32$ minutes. Since the marathon is 26.2 miles, he completes 8 3-mile cycles for a total of $32 \times 8 = 256$ minutes. He still has 26.2 - 24 = 2.2 miles to go. He has to run the next 2 miles which takes 16 minutes. 256 + 16 = 272. This leaves .2 miles for him to walk.
$.2 \times 16 = 3.2$
272 + 3.2 = 275.2 **Answer**

3. $5 - 7(5^2 - 3^3)^4 =$
$5 - 7(25 - 27)^4 =$
$5 - 7(-2)^4 = 5 - 7(16) =$
5 - 112 = -107 **Answer**

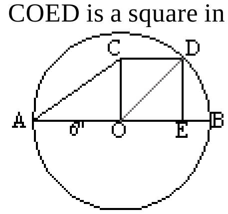
4. Chickens have two legs each and pigs have four. Both animals have a single head.
Let c = the number of chickens.
Let p = the number of pigs.
c + p = 50 (Eq. 1)
2c + 4p = 170 (Eq. 2)
2c + 2p = 100 (Eq. 3 = $2 \times$ Eq. 1)
2p = 70 (Eq. 2 - Eq. 3)
p = 35 **Answer**

5. $66\frac{2}{3}\% = \frac{2}{3}$
Alan's copy of the picture was $\frac{2}{3}$ the size of the original which he then gave to Beth. Beth reduced her copy to $66\frac{2}{3}\%$ of its current size or $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$ of the original.
To get the picture back to the original size we'll have to multiply by the reciprocal of $\frac{4}{9}$. $\frac{1}{\frac{4}{9}} = \frac{9}{4} = 2\frac{1}{4} = 225\%$ **Answer**

6. The trapezoid looks as follows:


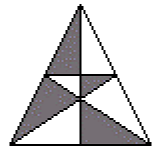
where x is the length of that piece of the base formed by dropping a perpendicular from the top corner of the trapezoid to the base (i.e., the height). To calculate the area, we must determine this height, h.
The perimeter of the trapezoid is 48 so
3x + 4x + 3x + 6x = 48
16x = 48
x = 3
$3^2 + h^2 = 9^2$
$9 + h^2 = 81$
$h^2 = 72$
$h = \sqrt{72} = 6\sqrt{2}$
The area of the rectangle formed by the two heights and the top of the trapezoid is
$12 \times 6\sqrt{2} = 72\sqrt{2}$
The area of the two triangles are:
$2 \times \frac{1}{2} \times 3 \times 6\sqrt{2} = 18\sqrt{2}$
$72\sqrt{2} + 18\sqrt{2} = 90\sqrt{2}$ **Answer**

7. $\frac{4!+3!}{3!+2!} = \frac{(4 \times 3 \times 2 \times 1) + (3 \times 2 \times 1)}{(3 \times 2 \times 1) + (2 \times 1)} =$
$\frac{(3 \times 2 \times 1)(4+1)}{(2 \times 1)(3+1)} = \frac{3 \times 5}{4} = \frac{15}{4} = 3.75$ **Answer**

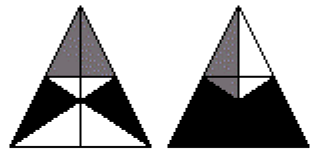
8. COED is a square in the picture below.


To determine AC, we must first determine OC. Draw the diagonal OD in square COED. OD is a radius of the circle. Therefore, its length is 6. If a side of the square is x, then
$x^2 + x^2 = 6^2$
$2x^2 = 36$
$x^2 = 18$
Let y = AC.
$y^2 = x^2 + 6^2 = 18 + 36 = 54$
$y = \sqrt{54} = 3\sqrt{6}$ **Answer**

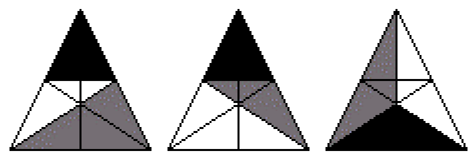
9. If the cube is made up of 125 smaller congruent cubes, then we have a $5\times5\times5$ cube. A cube has six sides. If we are to look for those cubes with at least 25% of their surface area painted, these must be cubes with $\frac{1}{4}\times 6 = 1\frac{1}{2}$ sides painted. But we can't have fractional sides painted here, so we're looking for all cubes that have at least 2 sides painted. The quickest way to determine this is to look at how many cubes have less than 2 sides painted. So, how may cubes have only one side painted? These are all the cubes on the outside of the larger cube that are not on an edge. There are $3\times 3 = 9$ of these on each surface or $9\times 6 = 54$. Now, how many don't have any of their surfaces painted? Only the ones in the interior of the cube. This is a $3\times 3\times 3$ inner cube or 27 more. 27 + 54 = 81
$\frac{125 - 81}{125} = \frac{44}{125}$ **Answer**
10. 2% of an hour = 0.02×60 minutes
= 1.2 minutes
30% of a minute is 0.3 minutes
1.2 minutes = 60 + 12 = 72 seconds
0.3 minutes = 18 seconds
72 - 18 = 54 **Answer**
11. $2^x \bullet 9^y$ = 2x9y
If 9 is a factor of 2x9y, then 2 + x + 9 + y = 11 + x + y must be a multiple of 9. Since 2 is a factor of 2x9y, y must be even (but not zero) as well. If 2 + x + 9 + y sums up to 18, x + y = 7. So, we can look at these (x,y) combinations: (5,2), (3,4), and (1,6). But we can eliminate the last 2 because $9^4 > 3000$.
$2^5 \bullet 9^2 = 32\times 81 = 2592$
$5\times 2 = 10$ **Answer**
(But, should we consider the sum of the digits being 27, which is the next multiple of 9? But if this were so, 27 - 11 = 16 which would give the combination (8,8) which is way too large.)
12. To find the units digit of $\left(133^{13}\right)^3$, we must first find the units digit of 133^{13}. If you look at the units digit for powers of 3,
3, 9, 27, 81, 243, 729, etc.
you will see that we have a series:
3, 9, 7, 1, 3, 9, 7, 1 ...
This repeats every 4 terms.
$\frac{13}{4}$ = 3 R1 so 133^{13} will end in a 3.
$3^3 = 27$ and the units digits of the final product will be 7. **Answer**
13. We need to determine how many triangles are in the figure. Start with triangles that have no lines within them.



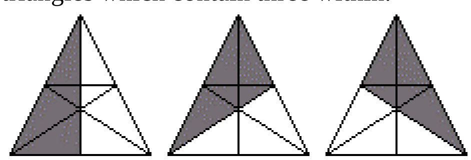
There are 8 triangles. Now look at those triangles with a single line within.



The first picture shows 3 (2 white and 1 gray) and the second shows 2 (1 white and 1 gray). Now look at triangles which contain two lines within.

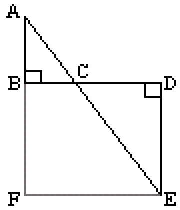


There are 2 in each of these 3 pictures (1 gray and 1 white). Similarly, look for triangles which contain three within.



We have 2 in the first (1 gray, 1 white) and 1 in each of the other two pictures (both gray). Finally, we have the whole triangle.
8 + 5 + 6 + 4 + 1 = 24 **Answer**
14. We can earn 0, 1, 3, 7 or 10 points with a shot. With one shot you can make:
1, 3, 7, or 10 points
With two shots you can make:
2, 6, 14, 20, 4, 8, 11, 13, and 17 points.
With three shots (using the values of 1 shot and 2 shots) you can make:
5 (3,1,1), 9 (3,3,3), 11 (1,3,7), 12 (10,1,1), 15 (7,7,1), 16 (10,3,3), 18 (10,7,1), 21 (10,10,1) or (7,7,7), 23 (10,10,3), 24 (10,7,7), 27 (10,10,7) and 30 (10,10,10)
The values we cannot obtain are 19, 22, 25, 26, 28, and 29. 6 **Answer**

15. In the diagram BD = 6, AB = 3 and DE = 5.



To determine AE, continue AB until it intersects with a line segment parallel to BD, which is EF. Then EF = BD = 6.
BF = DE = 5.
AF = AB + BF = 3 + 5 = 8
Let x = AE.
$6^2 + 8^2 = x^2$
$36 + 64 = 100 = x^2$
x = 10 **Answer**

16. The coordinates of one of the endpoints of a diagonal of a rectangle are (-4,2) and the coordinates of the point of intersection of the diagonals are (1,-1). The sides of the rectangle are parallel to the axes. The point of intersection of the diagonals is the midpoint of the diagonals. Therefore, the x coordinate of the endpoint of the diagonal that starts at (-4,2) is:
-4 + (1 - (-4)) + (1 - (-4)) = -4 + 5 + 5= 6.
The y coordinate is:
2 + (-1 - 2) + (-1 - 2) = 2 - 3 - 3 = -4.
So, the opposite endpoint of the diagonal is (6, -4). Since the sides of the rectangle are parallel to the axes, the other two points are (-4, -4) and (6, 2). Therefore, the sides of the rectangle are of sizes:
6 - (-4) = 10 and 2 - (-4) = 6.
$6 \times 10 = 60$ **Answer**

17. $2^{-1} + 2^{-2} + 2^{-3} + \ldots + 2^{-9} + 2^{-10} =$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{512} + \frac{1}{1024} =$$

$$\frac{3}{4} + \frac{1}{8} + \ldots + \frac{1}{512} + \frac{1}{1024} =$$

$$\frac{7}{8} + \ldots + \frac{1}{512} + \frac{1}{1024} = \ldots =$$

$\frac{1023}{1024}$ **Answer**

Note: This is a series of the form

$$\sum_{i=1}^{n} \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

18. $9a^2 - 8b^2 = 1800$

$$\frac{a}{b} = \frac{4}{3}$$

$$a = \frac{4}{3} b$$

$$9\left(\frac{4}{3}b\right)^2 - 8b^2 = 1800$$

$16b^2 - 8b^2 = 1800$
$8b^2 = 1800$
$b^2 = 225$
b = 15

$$a = \frac{4}{3} b = \frac{4}{3} \times 15 = 20$$

$ab = 20 \times 15 = 300$ **Answer**
Note: It is possible that a = -20 and b = -15 and we would get the same answer. However, it is not possible that a and b have different signs because their ratio is positive.

19. We must guarantee that we pick three cards of the same color. If we pick one card of the first color, then one of the second color, then one of the third color, repeat this, and then pick one of any color, we will definitely have three of some color. Any other order would give us three of some color, sooner.
$2 \times 3 + 1 = 7$ **Answer**

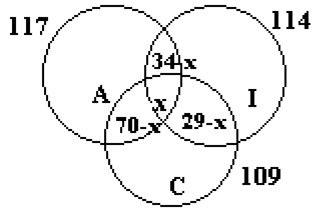
20. We need to find how many paths spell the word MATH when we are allowed to move to an adjacent letter to the right, left, up or down.

H
H T H
H T A T H
H T A **M** A T H
H T A T H
H T H
H

Start by moving from the M one step up to the A. From this A, we can get to 3 T's. The T right above the A can get to 3 H's, but the two T's adjacent to the A can only get to 2 H's each. Thus we have 3 + 2 + 2 paths. If you look at each of the other A's, you will observe that each of them has access to 3 T's which have access to 7 H's.
$7 \times 4 = 28$ **Answer**

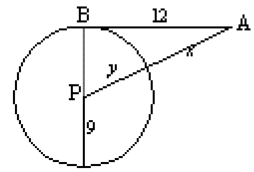
21. 164 of the 400 students do not take any of these courses. Therefore, 400 - 164 = 236 students take at least one. The requirements:
117 take algebra.
109 take advanced computer.
114 take industrial technology.
70 take algebra and advanced computer.
34 take algebra and industrial technology.
29 take both advanced computer and industrial technology.
Let x = the number of students taking all

three courses.
To make this easy to see, this information can be represented by the following illustration:



The circle marked A represents those students who take algebra. Just outside the circle is the number of students taking algebra. A is the number of students who take only algebra. We are told that 70 take both algebra and computing, but this includes students who also take industrial technology. Therefore, the intersection of A and C contains both those students who take algebra and computing (70-x), and those who take all three (x). Similarly, we have 34 students taking both industrial technology and algebra, but this must be broken down into 34 -x and x, as we did with the students taking algebra and computing.
Let us first determine how many students take two courses, but not the third.
117 = A + (34 - x) + (70 - x) + x
117 = A + 34 + 70 - x
A - x = 117 - 104 = 13 (Eq. 1)
Similarly,
114 = I + (34 - x) + (29 - x) + x
114 = I + 34 + 29 = x
I - x = 114 - 63 = 51 (Eq. 2)
And,
109 = C + 29 - x + 70 - x + x
109 = C + 29 + 70 - x
C - x = 109 - 99 = 10 (Eq. 3)
Now, let's add up all the kids represented in the diagram:
A + I + C + x + (34 - x) + (29 - x) + (70 - x) = 236
(A - x) + (I - x) + (C - x) + x + 34 + 29 + 70 = 236
Using Eq. 1, 2, and 3:
13 + 51 + 10 + x + 34 + 29 + 70 = 236
74 + x + 133 = 236
x + 207 = 236
x = 29 **Answer**

22. Draw a line from P to B as in the figure below.



Triangle ABP is a right triangle and PB = 9. Since we have two sides which are 9 and 12, AP = 15 (since this triangle is similar to a 3, 4, 5 right triangle). y is a radius of circle P which means y = 9. Therefore,
x = 15 - 9 = 6.
$\frac{x}{x+y} = \frac{6}{15} = \frac{2}{5}$ **Answer**

23. Melissa is caught after 60 seconds or 1 minute. At 55 mph, this means she has gone $\frac{55}{60} = \frac{11}{12}$ mile. The sports car started out $\frac{1}{2}$ mile before her but caught her $\frac{1}{2} + \frac{11}{12} = \frac{6}{12} + \frac{11}{12} = \frac{17}{12}$ miles later. This took the sports car 60 seconds.
$60 \times \frac{17}{12} = 5 \times 17 = 85$ **Answer**

24. We need to find how many ordered triples of three primes sum to 24. First, list the primes less than 24.
2, 3, 5, 7, 11, 13, 17, 19, 23
Start with 2; try each of the other primes and see if there is one that matches.
(2, 3, 19), (2, 5, 17), (2, 11, 11)
How about starting with 3? Never! If we don't use 2, we'll be adding 3 odd numbers which will always result in an odd number. Therefore, all we need do is figure out how many ordered triples we have. For (2, 3, 19) and (2, 5, 17), there are 6 versions for each (i.e., 3!). For (2, 11, 11) there are only 3 since half of the combinations will look exactly the same as the other half.
6 + 6 = 3 = 15 **Answer**

25. Since this is an isosceles triangle, two sides must be the same.
3x + 62 = 7x + 30
4x = 302
x = 8
7x + 30 = 5x = 50
2x = 20
x = 10
3x + 62 = 5x + 50
2x = 12
x = 6
Clearly, the smallest possible perimeter will

happen with the smallest sides, i.e., using the smallest value of x.
3x + 62 + 7x + 30 + 5x + 50 = 15x + 142 = 15 × 6 + 142 = 90 + 142 = 232 **Answer**

26. Janelle averages 40 km per hour on level ground. She averages 60% of that or .6 × 40 = 24 km per hour riding uphill. She averages 120% or 1.2 × 40 = 48 km per hour riding downhill.
It will take her $\frac{5}{40} = \frac{1}{8}$ hour to do 5 km on level ground. It will take her $\frac{6}{24} = \frac{1}{4}$ hour to do 6 km uphill and it will take her $\frac{6}{48} = \frac{1}{8}$ hour to do 6 km downhill. In $\frac{1}{8} + \frac{1}{4} + \frac{1}{8} = \frac{1}{2}$ hour, she travels 5 + 6 + 6 = 17 km. 17 × 2 = 34 **Answer**

27. The probability of drawing a red marble is $\frac{2}{5}$. The probability of drawing a green marble is $\frac{3}{7}$. Therefore, the probability of drawing a blue marble is
$1 - \left(\frac{2}{5} + \frac{3}{7}\right) = 1 - \left(\frac{14}{35} + \frac{15}{35}\right) =$
$1 - \frac{29}{35} = \frac{6}{35}$
Given that there are less that 50 marbles, there must only be 35 for these probabilities to hold. Thus, there are $\frac{2}{5} \times 35 = 14$ red marbles, $\frac{3}{7} \times 35 = 15$ green marbles and 6 blue ones.
$\frac{6}{35} \times \frac{5}{34} = \frac{3}{7} \times \frac{1}{17} = \frac{3}{119}$ **Answer**

28. Brent drinks $\frac{1}{3}$ cup of lemonade leaving $\frac{2}{3}$ cup. He then fills the cup back up with water and mixes. When he drinks the next $\frac{1}{3}$ cup, only $\frac{2}{3} - \frac{2}{3} \times \frac{1}{3} = \frac{6}{9} - \frac{2}{9} = \frac{4}{9}$ of the original glass of lemonade remains. After he fills it up again and drinks $\frac{1}{3}$ cup, only $\frac{4}{9} - \frac{4}{9} \times \frac{1}{3} = \frac{4}{9} - \frac{4}{27} = \frac{12}{27} - \frac{4}{27} = \frac{8}{27}$ of the original lemonade remains. $\frac{8}{27}$ **Answer**

29. Let abc be a three digit number. If a + b + c = 26, the numbers must be 9, 9, and 8, in no particular order. We are multiplying this number by 7, 11 and 13. 7 × 11 × 13 = 1001.
abc × 1001 = abcabc
Therefore, 9 must appear 4 times. 4 **Answer**

30. Both pumpkins together weigh 20 pounds, but we are told that Joe's pumpkin weighs more than his sister's. Therefore, his pumpkin must be greater than 10 pounds. His pumpkin costs $3.52.
352 = 2 × 2 × 2 × 2 × 2 × 11
If his pumpkin weighed 11 pounds, his sister's pumpkin would weigh 9. But 9 does not divide $0.48. Are there any other factors of 352 that are between 10 and 20? 16 is the other factor between 10 and 20. If this is the case, he would have paid $0.22 per pound for his pumpkin. His sister would have paid $0.22 - $0.10 = $0.12 per pound or $0.12 × (20 - 16) = $0.48 which is exactly what the problem stated she spent. Therefore, the weight of Joe's pumpkin is 16. **Answer**

<u>TARGET ROUND</u>

1. We need to create two prime numbers using the digits 2, 4, 5 and 7 each once. Any number that ends in 2 (with the exception of 2 itself) is divisible by 2, and therefore not prime. The same goes for 4, because it is even, and therefore, divisible by 2. 5 is similar to 2; any multiple-digit number that ends in 5 is not prime with the exception of 5 itself. Thus, one of our numbers must end in 7. Of two-digit numbers, only 47 is prime which means it is of no use, since the other number would also have to use 2 digits (and neither 52 or 25 is a prime). Therefore, we must find the smallest three digit prime made up of 4, 5, and 7. Well... how about 457? It's prime! 457 × 2 = 914 **Answer**

2. We are given an ordered triple (a, b, c) where each of the elements are different positive integers < 5. The total number of ordered triples is 4 × 3 × 2 = 24. How many ordered triples are there such that a < b < c? Suppose a = 4. There would be none, because b and c would have to be greater

than 4. How about a = 3? There also are none because while b =4, what would c be? So now we look at a = 2.
(2, 3, 4) works just fine. And a = 1? There are a few. (1, 2, 3), (1, 2, 4), (1, 3, 4). Therefore, we have 4 possible ordered triples.

$\frac{4}{24} = \frac{1}{6}$ **Answer**

3. Let x = the original rate of speed.

$\frac{120}{x}$ = the number of hours it takes to go 120 miles at x miles per hour.

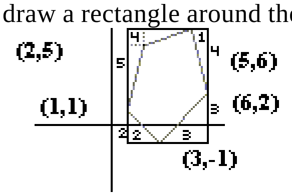
30 minutes = $\frac{1}{2}$ hour.

$\frac{120}{x+8} = \frac{120}{x} - \frac{1}{2}$

$\frac{120}{x+8} = \frac{240}{2x} - \frac{x}{2x} = \frac{240 - x}{2x}$

$(x + 8) \times (240 - x) = 120 \times 2x$
$240x - x^2 + 1920 - 8x = 240x$
$-x^2 + 232x + 1920 = 240x$
$x^2 + 8x - 1920 = 0$
Let's factor 1920:
$1920 = 2 \times 5 \times 192 =$
$2 \times 5 \times 2 \times 2 \times 48$
We need 2 factors that differ by 8. And it is evident that they are 40 and 48.
(x + 48)(x - 40) = 0
x = -48 (negative miles don't work here...)
x = 40 **Answer**

4. First graph the points, connect them, and draw a rectangle around the pentagon.

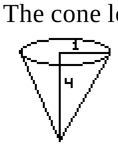


Note that where the points of vertices of the pentagon intersect the sides of the rectangle, the lengths of the pieces of the lines have been inserted. The sides of the rectangle are 7 and 5. Therefore, the area of the rectangle is $7 \times 5 = 35$. To get the area of the pentagon, we need to subtract the areas of the three triangles and one quadrilateral formed by drawing the rectangle. Start with the lower left hand triangle. It is a right triangle with sides of 2 and 2. Therefore, its area is $\frac{1}{2} \times 2 \times 2 = 2$. Similarly, the area of the lower right hand triangle is $\frac{1}{2} \times 3 \times 3 =$ 4.5. The area of the upper right hand triangle is $\frac{1}{2} \times 1 \times 4 = 2$. Now we deal with the upper left hand side. This is not a triangle, but a quadrilateral. We can break this quadrilateral into a square of sides 1, and two triangles. Look closely, and you can see that the "4" in the upper left hand corner has a dotted line surrounding it which represents the square. This creates two right triangles, one of sides 3 by 1, and one of sides 4 by 1. Thus, the area of the quadrilateral is:

$(1 \times 1) + (\frac{1}{2} \times 3 \times 1) + (\frac{1}{2} \times 4 \times 1) =$
1 + 1.5 + 2 = 4.5
Adding up all the areas outside of the pentagon, we get :
2 + 4.5 + 2 + 4.5 = 4 + 9 = 13
35 - 13 = 22 **Answer**

5. Set M consists of the positive odd integers less than 92 which are multiples of 3 or 5.
The odd multiples of 3 are 3, 9, 15, ..., 87.
$3 + 9 + ... + 81 + 87 = 90 \times 7.5 = 675$
The odd multiples of 5 are 5, 15, 25, ..., 85.
$5 + 15 + ... + 75 + 85 = 90 \times 4.5 = 405$
But, remember that the LCM of 5 and 3 is 15. Therefore, all odd multiples of 15 are in there twice.
15 + 45 + 75 = 135
675 + 405 - 135 = 1080 - 135 = 945 **Answer**

6. The cone looks like this:



$V_{cone} = \frac{1}{3}\Pi r^2 h = \frac{1}{3}\Pi(1^2)4 = \frac{4}{3}\Pi$

$V_{sphere} = \frac{4}{3}\Pi r^3 = \frac{4}{3}\Pi$

$r^3 = 1$; r = 1 **Answer**

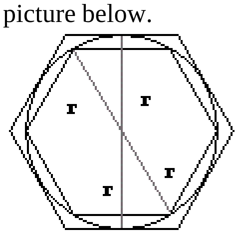
7. $(2^{4x+8})(4^{2x+3}) = 8^{2x+6}$
Get everything into powers of 2.
$(2^{4x+8})(2^{2(2x+3)}) = 2^{3(2x+6)}$
$(2^{4x+8})(2^{4x+6}) = 2^{6x+18}$
$2^{8x+14} = 2^{6x+18}$
8x + 14 = 6x + 18
2x = 4
x = 2 **Answer**

8. Look at the number of diagonals in a polygon starting with 4 sides:
2, 5, 9, 14, 20, 27, 35

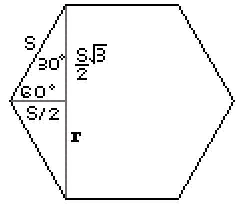
(note that the difference between the number of diagonals of polygons with sides n + 1 and n is one more than the difference between the number of diagonals of polygons of sides n and n - 1. It is immediately evident that 20 + 14 = 34. This corresponds to polygons with sides 7 and 8. 8 - 7 = 1 **Answer**

TEAM ROUND

1. The front of the train has travels for 40 seconds at 60 mph. At 60 mph, the train will travel 1 mile in 1 minute, or 60 seconds. Therefore, it travels $\frac{2}{3}$ mile in the 40 seconds. $\frac{2}{3} \times 5280 = 3520$ **Answer**
2. There are $5 \times 4 = 20$ ways that we can chose two different numbers. But, since we hold both down, choosing a and b is the same as choosing b and a. Therefore, we have only 10 choices. With those ten choices, now come 5 different ways to pick the single button. $10 \times 5 = 50$ **Answer**
3. Alexandria had the letters R, O, O, V, G, N, and O. Normally, there would be 7! different ways to make a 7-letter arrangement. But since we have 3 letters that are the same, we will have 3! of each 7-letter arrangement. (Example: suppose you were to make ROOVGNO -- to differentiate the three different O's, call them O_1, O_2, and O_3. So the six different versions of ROOVGNO are:
   1: $RO_1O_2VGNO_3$
   2: $RO_1O_3VGNO_2$
   3: $RO_2O_1VGNO_3$
   4: $RO_2O_3VGNO_1$
   5: $RO_3O_1VGNO_2$
   6: $RO_3O_2VGNO_1$
   $\frac{7!}{3!} = 7 \times 6 \times 5 \times 4 = 42 \times 20 = 840$ **Answer**
4. We are given a circle inscribed in a larger regular hexagon and a smaller regular hexagon inscribed in the circle, as in the picture below.



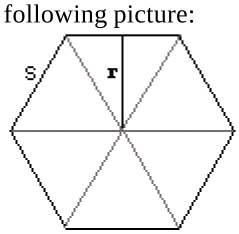
Two diameters, with radius, r, have been drawn in the circle Let's start by determining the area of the larger hexagon. We can move the vertical diameter over so that it connects 2 of the vertices and we know that its length is 2r.



To find the side of the hexagon, we draw a perpendicular from a vertex to the diagonal. This bisects the diameter. Since a hexagon has interior angles of 120°, the perpendicular also bisects the interior angle giving us a 30°, 60°, 90° right triangle. If s is the side of the hexagon, then the side opposite the 60° angle is $\frac{s\sqrt{3}}{2}$. But this is also half the diameter or r.

$r = \frac{s\sqrt{3}}{2}$

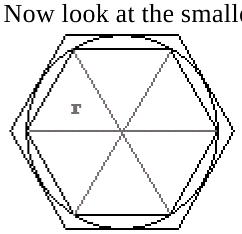
The area of the hexagon is equal to the area of the 6 equilateral triangles as shown in the following picture:



The triangles have a height of r and a base of s. Thus, the area of the hexagon =

$$6 \times \left(\frac{1}{2} rs\right) = 3rs = 3s\left(\frac{s\sqrt{3}}{2}\right) = \frac{3s^2\sqrt{3}}{2}$$

Now look at the smaller hexagon:



From the first illustration, we know that these are all diameters of the circle with length r. And we also know that we have 6 equilateral triangles of length r. Thus, the area of the smaller hexagon is:

$$6 \times \left(\frac{1}{2} \times \frac{r\sqrt{3}}{2} \times r\right) = \frac{3r^2\sqrt{3}}{2}$$

Substituting for r:

$$\frac{3\left(\frac{s\sqrt{3}}{2}\right)^2\sqrt{3}}{2} = \frac{\frac{9s^2\sqrt{3}}{4}}{2} = \frac{9s^2\sqrt{3}}{8}$$

The ratio of the area of the smaller hexagon to the larger is:

$$\frac{\frac{9s^2\sqrt{3}}{8}}{\frac{3s^2\sqrt{3}}{2}} = \frac{\frac{9}{8}}{\frac{3}{2}} = \frac{9}{8} \times \frac{2}{3} = \frac{3}{4}$$ **Answer**

5. Choosing m people from a group of n people can be determined by the equation:

$$\frac{n!}{m! \times (n-m)!}$$

In this case, we need to look at all the ways we can choose 6 from 11, 7 from 11, and so on.

$$\frac{11!}{6!5!} = \frac{11 \times 10 \times 9 \times 8 \times 7}{5 \times 4 \times 3 \times 2 \times 1} =$$

$11 \times 3 \times 2 \times 7 = 462$

$$\frac{11!}{7!4!} = \frac{11 \times 10 \times 9 \times 8}{4 \times 3 \times 2 \times 1} = 11 \times 10 \times 3 = 330$$

$$\frac{11!}{8!3!} = \frac{11 \times 10 \times 9}{3 \times 2 \times 1} = 11 \times 5 \times 3 = 165$$

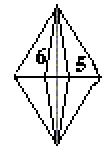
$$\frac{11!}{9!2!} = \frac{11 \times 10}{2 \times 1} = 11 \times 5 = 55$$

$$\frac{11!}{10!1!} = \frac{11}{1} = 11$$

And there is only 1 way of choosing 11.
$462 + 330 + 165 + 55 + 11 = 1 =$
1024 **Answer**

6. 31x + 29y = 1125
29(x + y) + 2x = 1125
2x is always even. Therefore, 29(x + y) must be odd and (x + y) must be odd. What is the maximum value of 29(x + y)?
$29 \times 38 = 1102$ Therefore, $x + y \leq 38$. But remember that x + y must be odd so $x + y \leq 37$. Suppose x + y = 37. Then 29(37) + 2x = 1125
1073 + 2x = 1125
2x = 52
x = 26; y = 11
Suppose x + y = 35. Then
29(35) + 2x = 1125
1015 + 2x = 1125
2x = 110
x = 55, but x and y are positive and the sum can't be greater than 37. If we choose any smaller value than 37 for x + y, we'll violate the conditions. 37 **Answer**

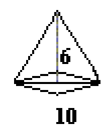
7. When the triangle is created and rotated around the x axis, it looks like this:



Literally it is made up of two cones lying on their sides and touching at their bases. The height of each cone is 5 and the radius is 6. The volume of one cone is:

$$\frac{1}{3}\Pi r^2 h = \frac{1}{3}\Pi\left(6^2\right)5 = 60\Pi$$

The volume of two cones is 120Π.
Now let's look at the triangle rotated around the y axis.



This time we have a single cone with a diameter of 10 (or radius of 5) and height of 6. Its volume is:

$$\frac{1}{3}\Pi r^2 h = \frac{1}{3}\Pi\left(5^2\right)6 = 50\Pi$$

$120\Pi - 50\Pi = 70\Pi$ **Answer**

8. If all 25 were right, the number would be divisible by 1 through 25. This would mean that the large positive integer would be a multiple of the LCM of 1 through 25. The LCM of 1 through 25 is:
$1 \times 2 \times 3 \times 2 \times 5 \times 7 \times 2 \times 3 \times 11 \times 13 \times 2 \times 17 \times 19 \times 23 \times 5$
But if 2 are wrong there must be 2 of these values by which the number is not divisible. The LCM created above was done by starting with 1, increasing each number by 1, etc., and looking at whether we have all the factors necessary to create the number. I.e., the 1 was for 1; the 2 for 2; the 3 for 3; the next 2 for 4 (since we already had a 2) etc. Therefore, to find the two adjacent values (and the largest ones), we work backwards. 25 required a 5, but 24 didn't require anything; we already had the factors. 23 is there, but 22, 21 and 20 already had the factors. 19 is there but 18 already had the factors. 17 is there and so is 2 (we needed an extra one to make 16). And now we have two adjacent factors and they

are 16 and 17. Let's remove 2 and 17. We get $1 \times 2 \times 3 \times 2 \times 5 \times 7 \times 2 \times 3 \times 11 \times 13 \times 19 \times 23 \times 5$
The product of all these numbers is 787,386,000. **Answer**

9. The number of combinations of 3 people chosen from 5 people is:
$$\frac{5!}{3! \times 2!} = \frac{5 \times 4}{2} = 10$$
First let's figure out how many combinations have two women and one man. Let's just list them:
$M_1F_1F_2$, $M_2F_1F_2$, $M_3F_1F_2$
There are 3 out of the 10 combinations which have two women and one man. In such a group, the probability that a woman is a chairperson is $\frac{2}{3}$. Therefore, the probability that the group is composed of two women and one man and that the chairperson is a woman is:
$\frac{3}{10} \times \frac{2}{3} = \frac{2}{10} = \frac{1}{5}$ **Answer**

10. Since Big Ben is a 12-hour clock, let's first deal with 12 hours. Then we can multiply by 2. In one hour, not counting the number of notes it strikes for the hour, Big Ben will strike 4 + 8 + 12 + 16 = 40 notes. Therefore, in a 12-hour period, Big Ben will strike:
$(40 \times 12) + (1 + 2 + ... + 11 + 12) =$
$480 + (13 \times 6) = 480 + 78 = 558$
For 24 hours worth:
$558 \times 2 = 1116$ **Answer**