Sprint Round

1. 0 (ways)
2. 2 (pieces)
3. 19
4. 210 (cubic centimeters)
5. 864 (eggs)
6. 2
7. $\frac{1}{3}$
8. 100 (cubes)
9. -5
10. 4 (edges)
11. 30
12. $\frac{3}{5}$
13. $\frac{\sqrt{2}}{2}$
14. 114 (students)
15. 90 (seams)
16. 24

Sprint Round

17. 12 (triangles)

18. 20 (miles)

19. 70 (square centimeters)

20. 484

21. $\frac{5}{18}$

22. -150

23. $\frac{3}{2}$

24. 15 (partitions)

25. 81

26. $\frac{9}{10}$

27. 5

28. 17

29. 176 (minutes)

30. 84 (square feet)

Target Round

1. 10 (minutes)

2. 8

3. 0.098

4. $\frac{3}{7}$

5. 12 (chords)

6. 194 (square centimeters)

7. 6720 (subcommittees)

8. 727

Team Round

1. 10 (degrees)

2. 66 (square units)

3. 56.5 (minutes)

4. 481 (black tiles)

5. $\frac{1}{15}$

6. $\frac{7}{6}$

7. 85

8. 17.3 (feet)

9. 0.73

10. 26 (triangles)

Countdown Round

1. 14.85 (dollars)
2. 110 (miles)
3. $4\frac{1}{2}$ (feet)
4. $\frac{7}{12}$
5. 31
6. 66 (sets)
7. 3
8. 462 (square feet)
9. 9
10. 19
11. 100 (days)
12. $-\frac{3}{2}$
13. 92 (points)
14. 60 (miles per hour)
15. 6 (hours)
16. $\frac{1}{12}$
17. 44 (percent)
18. -4
19. 36
20. $\frac{1}{22}$
21. 121
22. 315 (tiles)
23. 18
24. 20 (triangles)
25. 15 (integers)
26. $9:20$
27. 700
28. $\frac{1}{5}$
29. 6
30. 36 (percent)
31. 100 (meters)
32. 128
33. 27 (square centimeters)
34. 56 (inches)
35. 1 (integer)
36. 4
37. $\frac{11}{16}$
38. 600
39. 12 (numbers)
40. 49 (square inches)
41. 400 (dollars)
42. 4 (integers)
43. 12 (inches)
44. 9
45. 27 (cubic centimeters)

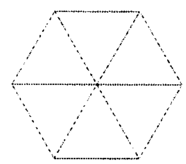
Countdown Round

46. 13 (dollars)
47. 45
48. 12,500 (zip codes)
49. 4.8 (miles per hour)
50. 11
51. $\frac{1}{3}$
52. 60 (degrees)
53. 15
54. 0
55. 9
56. 41
57. 192 (centimeters)
58. 14
59. 0
60. 9 (cubic inches)
61. 17 (times)
62. 64 (square inches)
63. 0
64. 26 (centimeters)
65. 1.4×10^7
66. 24 (squirrels)
67. 1320 (ways)
68. 200 (percent)
69. 8000
70. $\frac{1}{4}$
71. $\frac{4}{9}$
72. 3 (integers)
73. 18
74. 55
75. 216 (cubic meters)
76. 105
77. 5 (diagonals)
78. 133
79. $\frac{13}{16}$
80. $\frac{2}{3}$

2001 State Masters Round – Answers

1.

(a) Since the opposing sides of H_2 are 2 units apart, every point on each line in the diagram below lies exactly one unit from two opposing sides. The only point that lies exactly one unit from every side is the intersection of the grid lines. So there is only one interior point that lies an integer number of units from every side of H_2.



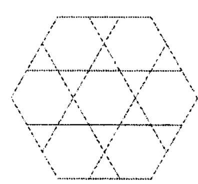
(b) The only hexagonal string for H_2 is (1,1,1,1,1,1).

(c) Each triangle in the diagram is isosceles and the height of the triangle is 1 and so the length of each side is $\frac{2\sqrt{3}}{3}$ and this is the length of the side of H_2. This follows from the Pythagorean Theorem.

(d) There are no points that lie an integer distance from exactly four sides. If there were such a point, it would have to lie on the intersection of exactly two of the grid lines.

2.

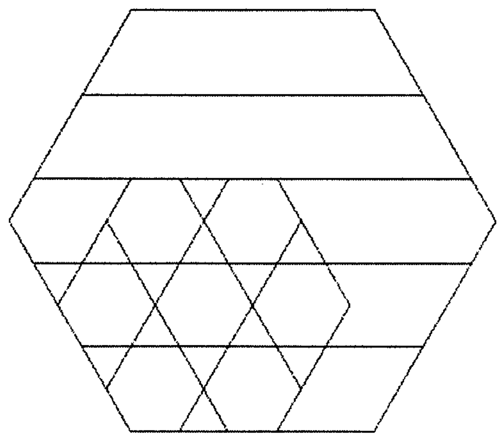
(a) The grid lines in the diagram below are fixed at integer distances from the sides of H_3. That is, every point on each grid line is an integer number of points from two opposing sides of H_3. There are no points that lie in integer distance from each side of H_3. If there were such a point it would have to lie on the intersection of exactly three grid lines.



(b) The triangle from the center of H_3 to any two adjacent vertices of H_3 is isosceles and the height of the triangle is 1.5 and so the length of each side is $\sqrt{3}$ and this is the length of each side of H_3. This follows from the Pythagorean Theorem.

(c) There are 12 interior points of H_3 that lie an integer distance from exactly four sides of H_3. Each such point must be the intersection of exactly two grid lines and there are clearly 12 such points.

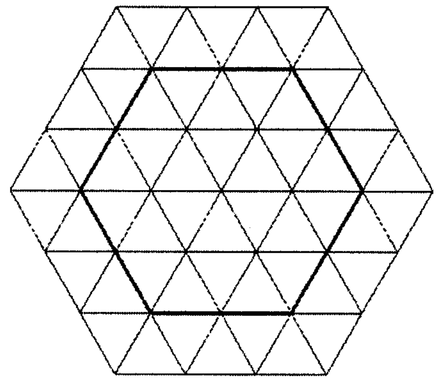
3. Embed H_3 in the interior of H_5 at any vertex. See the diagram.



H_3 can be placed at any vertex. Any point in H_5 that is an integer distance from any side of H_5 will be inside of a copy of H_3 and will also be in integer distance from a parallel or coincident side of H_3. If a point inside of H_5 is an integer distance from every side of H_5 then that point will be inside a copy of H_3 and an integer distance from each side of H_3. We already know this is impossible and so there can be no interior points of H_5 an integer distance from every side of H_5.

4. Clearly there are no interior points of H_1 that are an integer distance from each side of H_1 and parts 2 and 3 establish this fact for n = 3 and 5. Let n = 2k-1 for k a natural number. Let P(k) be the statement that H_{2k-1} has no interior points that lie an integer distance from each side of H_{2k-1}. We have that P(1),P(2), and P(3) are all true. Suppose P(k) is also true. H_n has no interior points that lie an integer distance from each side of H_n where n = 2k-1. Consider H_{n+2}. Note that n+2 = (2k-1) + 2 = 2(k+1)-1. As in problem 3, H_n embeds in H_{n+2} at each vertex and if P(k+1) were a false statement then P(k) would have to be false and that is a contradiction. Therefore, by the principle of mathematical induction the statement P(k) is true for all natural numbers k.

5. Using the same idea in problem 3 it is clear that H_2 embeds in H_4 at every vertex and H_{2k} embeds in H_{2k+2} for all k. So if n = 2k, H_{2k+2} will have all of the interior points of H_{2k} that lie an integer distance from each side of H_{2k} and all we need do is count the additional interior points of H_{2k+2} that lie an integer distance from each side of H_{2k+2} . It is easiest to do this by embedding H_{2k} inside of H_{2k+2} directly at the center (see the diagram).



The number of interior points in H_{n+2} that lie an integer distance from each side of H_{n+2} is the sum of the number of interior points of H_n an integer distance from each side of H_{n+2} and the number of boundary points of H_n an integer distance from each side of H_{n+2}. The number of boundary points of H_n is given by 3n Further, it can be deduced that the number of interior points of H_n that lie an integer distance from each side of H_n is given by $\frac{3(n-1)^2+1}{4}$ for n = 2, 4 and 6. Let n = 2k and P(k) be the statement that the number of interior points of H_n that lie an integer distance from each side of H_n is given by $\frac{3(n-1)^2+1}{4}$. It is easy to verify that P(1) is true. Suppose P(k) is true. By the way H_n embeds in H_{n+2} at the center, the number of interior points of H_{n+2} that lie an integer distance from each side is $\frac{3(n-1)^2+1}{4} + 3n$. It is an algebra exercise to reduce this to the form $\frac{3(n+1)^2+1}{4} = \frac{3((n+2)-1)^2+1}{4}$.

6. It is clear that the numbers in a hexagonal string for H_n must appear in pairs that sum to n where n is even. The string (1,1,1,3,3,3) satisfies this condition for H_4 but fails to be a hexagonal string because no point in H_4 is 1 unit from 3 sides of H_4. The string (1,1,3,3,5,5) meets the necessary condition for n = 6. Further, since the potential point has two ones in the string the point lies on a diagonal of H_6 and this requires the other two numbers to be 3 so this is a hexagonal string. The string (1,3,3,3,3,5) meets the necessary condition to be a hexagonal string for H_6, but it is not a hexagonal string because of the 4 threes. No point in any H_n for n even lies the same integer distance from exactly 4 sides. This follows by induction from 1d.

7. If (a,b,c,d,e,f) is a hexagonal string then n must be even. Further, the numbers must appear in pairs that sum to n. So if n = 2k then a+b+c+d+e+f = 3(2k) so it must be divisible by 6.