WARM-UPS

The Warm-Ups contain problems that generally survey the middle school mathematics curriculum. For use in the classroom, the problems in the Warm-Ups serve as excellent additional practice for the mathematics that students are already learning. In preparation for competition, the Warm-Ups can be used to prepare students for problems that they will encounter in the Sprint Round.

Answers to the Warm-Ups include one-letter codes, in parentheses, indicating appropriate problem-solving strategies. However, students should be encouraged to find alternative methods of solving the problems; their methods may be better than the one provided! The following strategies are used: **C** (Compute), **F** (Formula), **M** (Model/Diagram), **T** (Table/Chart/List), **G** (Guess & Check), **S** (Simpler Case), **E** (Eliminate) and **P** (Patterns).

MATHCOUNTS Symbols and Notation

Standard abbreviations have been used for units of measure. Complete words or symbols are also acceptable. Square units or cube units may be expressed as units^2 or units^3.

WARM-UP 1

(For #1 - #4) Glacier Park Lodge was built in 1912 in Montana. Sixty columns support the verandas and form a colonnade in the lobby. Each column is made from a gigantic fir or cedar tree 500 to 800 years old that still retains its bark. Mule teams dragged tree trunks from the railhead to the building site. Each has a diameter of 36 to 42 inches, a height of 40 feet and a weight of 15 tons.

1. _____ Each flat car on a train held two of these columns. What was the total weight, in pounds, of the two columns loaded on a flat car?

2. _____ What is the minimum circumference of a column rounded to the nearest inch?

3. _____ What is the maximum circumference of a column rounded to the nearest inch?

4. _____ A representative column has diameter 42 inches, height 40 feet and weight 15 tons. What is the mean number of pounds per cubic foot? Round your answer to the nearest whole number.

5. _____ What is the product of all the even integers from –6 to 7, inclusive?

6. _____ Rick has 6 different pairs of socks. What's the probability that two randomly selected socks will be from a matching pair? Express your answer as a common fraction.

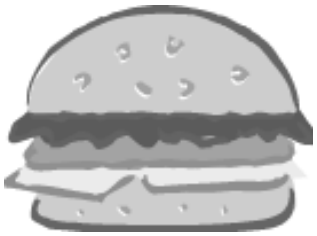
7. _____ For what single digit value of n is the number n5,3nn,672 divisible by 11?

8. _____ Suppose that a * b = a + b + ab. If x * 1 = 5, what is the value of x?

9. _____ Mrs. Smale's class filled out a survey. Here are the results for her 30 students.

14 like hot dogs
18 like cheeseburgers
16 like tacos
8 like both hot dogs and cheeseburgers
7 like tacos and cheeseburgers
6 like tacos and hot dogs
1 likes none of these

How many students like all three?



10. _____ What is the sum of the positive integer factors of 225?

WARM-UP 1

Answers

1.	60,000	(C)	5.	0	(C, P,T)	8.	2	(C, F)
2.	113	(F)	6.	$\frac{1}{11}$	(F, T)	9.	2	(M, G,F)
3.	132	(F)	7.	3	(E, P)	10.	403	(T, P,F)
4.	78	(F)						

Solution — Problem #6

Imagine Rick picking the socks one at the time. For his first pick, he can choose any sock, and still have an equal chance of getting a match on his second pick. So his first pick really doesn't enter into the answer. For his second pick, he has 11 socks left to choose from, only one of which will make a match. So the probability of picking the other sock that will make the match is 1/11.

Connection to... Tests for divisibility (Problem #7)

You probably already know a few "tricks" or divisibility rules for numbers like 2, 3, 4 and 5. To see if a number is divisible by 3, you just have to add all of the digits of the number together and see if the sum of the digits is divisible by 3. There is a similar test for divisibility by 11. We can calculate a number's *alternating digit sum* and see if that sum is divisible by 11. For example, to see if the number 25,949 is divisible by 11, we can alternate putting subtraction and addition signs between the digits, always starting by putting a subtraction sign after the first digit: 2-5+9-4+9. This alternating digit sum comes out to 11, which is divisible by 11, so 25,949 is divisible by 11. For #7, we would get n - 5 + 3 - n + n - 6 + 7 - 2. By simplifying the expression to n - 3, you can find the value(s) for n which would make the alternating digit sum divisible by 11. Remember zero is divisible by 11. How many other numbers do you know divisibility tests for?

Investigation & Exploration (Problem #10)

To understand the factors of 225, consider its prime factorization: $5^2 \cdot 3^2$. All of the factors of 225 can now be viewed in a table by making row and column headings out of the breakdown of the two unique prime factor parts (5^2 and 3^2). To fill in the table, multiply the corresponding row and column headers together. The interior of the table will show all of the factors of 225:

	1	**3**	**3^2**
1	1	3	9
5	5	15	45
5^2	25	75	225

Using this method, try to find all of the factors of 72. How is finding the number of factors in 4500 the same and different? How many different, positive factors does 4500 have? Can you come up with a method for determining how many different, positive factors a number has if you are given its prime factorization?

WARM-UP 2

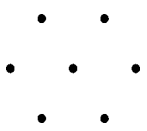
1. _____ The skull of a Tyrannosaurus Rex found in 1990 weighed a ton and was 5 feet long. What was the mean number of pounds per linear foot?

2. _____ For a recent year, the average wind speed, in miles per hour, for the ten windiest U.S. cities are listed below:

Amarillo, Texas 13.5	Blue Hill, Massachusetts 15.4
Boston, Massachusetts 12.5	Casper, Wyoming 12.9
Cheyenne, Wyoming 12.9	Dodge City, Kansas 14.0
Goodland, Kansas 12.6	Great Falls, Montana 12.7
Lubbock, Texas 12.4	Rochester, Minnesota 13.1

What is the median of the average wind speeds, in miles per hour, for these ten cities? Express your answer as a decimal to the nearest tenth.

3. _____ In the hexagonal lattice shown to the right, each point is one unit from its nearest neighbor. How many equilateral triangles can be drawn using a combination of three of the lattice points as vertices?



4. _____ The number 1 is both a smute and thripe. If the integer s is a smute, then the next smute is s + 5. If the integer t is a thripe, then the next thripe is 2t + 1. What is the smallest whole number greater than 1, that is both a smute and a thripe?

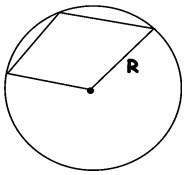
5. _____ In 1967, about 900 eagles were believed to exist in the continental 48 states. More than 200,000 are now believed to be present. What is the smallest whole number factor that the number of eagles in 1967 could have been multiplied by to yield more than 200,000 eagles now?

6. _____ The sum of the squares of two consecutive positive integers is 85. What is the sum of the two integers?

7. _____ Dave and Nick share their bread with Albert. Dave has 5 loaves of bread and Nick has 3 loaves. They share the bread equally among the three of them. Albert gives Dave and Nick $8, which they agree to share in proportion to the amount of bread they each gave away. How many dollars should Dave receive?

8. _____ Let x be a positive number and y be its reciprocal. Compute $\frac{1}{x+1}+\frac{1}{y+1}$.

9. _____ A rhombus is formed by two chords and two radii of a circle with radius R meters. What is the number of square meters in the area of the rhombus? Express your answer as a common fraction in terms of R.



10. _____ In a bag, there are 3 red marbles and B blue marbles. Two marbles are randomly selected from the bag without replacement. The probability that the two marbles are the same color is 0.5. Calculate the sum of all possible values of B.

WARM-UP 2

Answers

1.	400	(C)	5.	223	(C)	8.	1	(C, S)
2.	12.9	(C)	6.	13	(G, F, E)	9.	$(R^2\sqrt{3})/2$	(F)
3.	8	(P)	7.	7	(T, E)	10.	7	(P, G)
4.	31	(P, T)						

Solution — Problem #8

This problem can be handled with a basic knowledge of algebra and fractions. Notice the progression of the expression if we just substitute 1/x for y and carry out all of the addition:

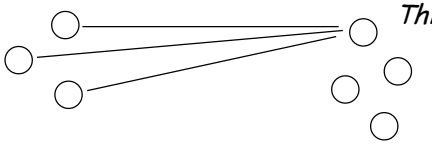
$$\frac{1}{x+1}+\frac{1}{y+1}=\frac{1}{x+1}+\frac{1}{\frac{1}{x}+1}=\frac{1}{x+1}+\frac{1}{\frac{1}{x}+\frac{x}{x}}=\frac{1}{x+1}+\frac{1}{\frac{x+1}{x}}=\frac{1}{x+1}+\frac{x}{x+1}=\frac{x+1}{x+1}=1$$

Seems like there must be an easier way...and there is! Make a simpler case out of the problem. The problem has to work for any x and y that fit the requirements. Therefore, let's pick x = 2 and y = 1/2. Plugging these values into the expression will lead to a much simpler solving process...especially if you are able to use a calculator!

$$\frac{1}{2+1}+\frac{1}{\frac{1}{2}+1}=\frac{1}{3}+\frac{1}{\frac{3}{2}}=\frac{1}{3}+\frac{2}{3}=1$$

Solution — Problem #10

3 Red marbles **B Blue marbles**



This demonstrates three ways to pick different colored marbles if B=1.

Each N represents the number of equally likely occurences of the events.

N_s(same color) = N_d(different colors) *Since the probability of pulling two the same color is .5.*

N_r(both Red) + N_b(both Blue) = N_d(different colors)

$3 + {}_{B}C_{2} = 3\cdot B$ *For every blue marble, there would be 3 red marbles that could be paired with it. See picture above.*

$3 + \frac{B(B-1)}{2} = 3B$

$6 + B(B-1) = 6B$

$B^2 - 7B + 6 = 0$ and then by trial and error, or by factoring, B=1 or B=6.

Sum = 1 + 6 = 7.

Investigation & Exploration (Problem #4)

Make a list of the first seven smutes. Do you see a pattern? Can you find a formula, in terms of x, so that you could find the x^{th} smute number? Make a list of the first seven thripes. What is a pattern that you see in this list of numbers? Again, can you develop a formula, in terms of x, so that you could find the x^{th} thripe number? Will any other numbers be both smutes and thripes?

WARM-UP 3

1. _____ The stem-and-leaf plot below reports the number of months spent on 20 separate investigations spanning the 1980's and 1990's. Find the positive difference between the median number of months spent on investigations during the 1980's and during the 1990's. Express your answer as a decimal to the nearest tenth.

Number of Months Spent on Separate Investigations

1980's		1990's
9 8 7	0	
7 5 4 3 2 1 1 0	1	0 5 5 9
	2	1 3
0	3	
	4	8 9

2. _____ The slope of the line tangent to the graph of $y = x^2$ at the point (3,9) is 6. What is the y-intercept of this tangent line?

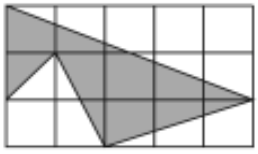
3. _____ A natural gas pipeline ruptured, triggering an explosion. The amount of soil required to fill the hole made by the explosion was equivalent to the amount of soil in a rectangular prism 86 feet long, 46 feet wide and 21 feet deep. How many dump truck loads, of 20 cubic yards of soil each, were needed to transport the soil?

4. _____ Alex has four bags of candy, weighing 1, 2, 3 and 4 pounds. To arrange them, he will pick up two bags which are next to one another, compare their weights on a balance scale and put them back, with the heavier one to the left. What is the maximum number of times he could possibly have to swap two bags before he has the bags in order from heaviest to lightest?

5. _____ The cube of the three-digit natural number A7B is 108,531,333. What is A+B?

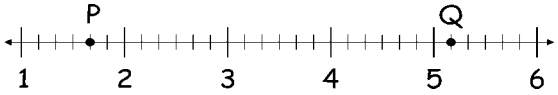
6. _____ A fraction is equivalent to 3/5. Its denominator is 60 more than its numerator. What is the numerator of this fraction?

7. _____ Each small square in the figure to the right has an area of 6 cm^2. What is the number of square centimeters in the area of the shaded region?



8. _____ "A plumber and his helper leave the shop at 8:20 A.M. to repair a faucet. They return at 11:10 A.M. They charge 60 cents an hour for the time the plumber is gone from the shop, half as much for his helper's time, and 85 cents for all of the materials" (paraphrased from The Thorndike's Arithmetics – Book Three, C. Thorndike, 1917). What was the number of dollars charged? Express your answer as a decimal to the nearest hundredth.

9. _____ What is the number of units in the length of segment PQ? Express your answer as a mixed number.



10. _____ What is the smallest positive integer that has 8, 30 and 54 as factors?

WARM-UP 3

Answers

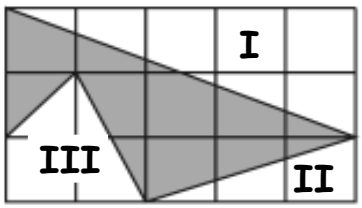
1.	8.5	(C)	5.	11	(E, S)	8.	3.40	(C, T)
2.	-9	(F, M)	6.	90	(P, G, C)	9.	$3\frac{1}{2}$	(C)
3.	154	(F)	7.	36	(F)	10.	1080	(P, T, C)
4.	6	(M, P)						

Solution — Problem #7

Sometimes the easiest way to find the answer to a problem is to find the answer to a different, but related, problem first. In this case, it may be easier to find the areas of the non-shaded regions and subtract that from the entire region. These non-shaded regions will help us to answer the question about the shaded region.

The **area of the entire figure** is 3 x 5 x 6 = **90 cm²**. (Remember there are just 15 square regions, but each of those has an area of 6 cm².)

The **area of the three non-shaded regions** are:



I. the right triangle at the top: 0.5 x 5 x 2 x 6 = **30 cm²**.

II. the right triangle at the bottom, on the right: 0.5 x 1 x 3 x 6 = **9 cm²**.

III. the region at the bottom, on the left (one square + one half-square + one right triangle): 6 + 3 + 0.5 x 1 x 2 x 6 = **15 cm²**.

Therefore, the **area of the shaded region** is 90 – 30 – 9 – 15 = **36 cm²**.

Connection to... Statistics (Problem #1)

In statistics there are a few different values we can use that are all considered *measures of central tendency*. The measure of central tendency that we are using in this problem is the median, or the middle number once the data has been ordered. Probably the most common measure of central tendency is the mean or the average. Many students are very familiar with how to find the average of a data set. What about the mode of a data set? How would you explain to someone how to find the mode? In a perfect bell curve, the median, mean and mode are all the same value. Can you create a data set where these three measures of central tendency are all the same?

Investigation & Exploration (Problem #4)

Programmers often need to sort a list of objects, whether numbers, names or some other set of information. A bubble sort, the type of sort described in this problem, systematically goes through the data comparing only two adjacent values, swapping the values if necessary and then starting the check again at the beginning until no more swaps can be done. It is called a bubble sort because the larger values are supposed to slowly "bubble up" toward the top of the list. This kind of sort is very time-consuming when there are a lot of data to sort. Can you create a formula that will tell you how many swaps will be needed in a worst-case scenario for n objects, meaning n objects are currently sorted in the exact opposite order of how you want them arranged?

Computer scientists have discovered other methods of sorting that are much more efficient. Explore what some of these other methods of sorting are and how they work.

WARM-UP 4

1. _____ Given that $4a+5b+7c = 13$ and $4a+3b+c = 19$, what is the value of $a+b+c$?

2. _____ During an 8-month Dungeoness crab season, 325 crabbers in Newport, Oregon landed 4,913,977 pounds of crab and sold them for $2.00 per pound. What was the average number of dollars earned by each crabber per month during the 8-month crab season? Express your answer to the nearest dollar.

3. _____ Jane invests $5000. She expects to have $\$5000(1.06)^8$ after eight years. Her interest rate, compounded annually, is $n\%$. What is the value of n?

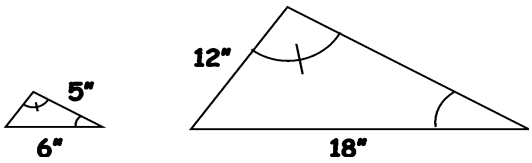
4. _____ How many factors of 1800 are multiples of 10?

5. _____ Two cards are randomly selected without replacement from a set of four cards numbered 2, 3, 4 and 5. What is the probability that the sum of the numbers on the two cards selected is 7? Express your answer as a common fraction.

6. _____ Point P is on the number line. The distance between zero and P is four times the distance between P and 30. What is the sum of the two possible values for P?

7. _____ How many digits are printed by a printer that prints all the whole numbers from 1 to 728, inclusive?

8. _____ What is the number of inches in the sum of the perimeters for the two similar triangles shown?



9. _____ The vertices of a triangle are (0,0), (0,y) and (x,0), where both x and y are greater than zero. The area of the triangle is 30 square units and the perimeter is also 30 units. What is the value of x + y?

10. _____ Yesterday, 2000 circular jellyfish, each of diameter 2 feet, were lying within a section of ocean floor that measured 120' by 360'. None of the jellyfish were overlapping each other. What percent of that section of ocean floor did the jellyfish cover? Express your answer to the nearest whole number.

WARM-UP 4

Answers

1.	4	(G, P)	5.	1/3	(F, T)	8.	60	(F, P)
2.	3780	(F)	6.	64	(M, G)	9.	17	(M, F, G)
3.	6	(F)	7.	2076	(T, P)	10.	15	(F)
4.	18	(P, E)						

Solution — Problem #4

Solution 1: Each factor that is a multiple of 10 must have at least one factor of 2 and one factor of 5. There are three factors with at least one factor of 2 (2,4,8). There are two factors with at least one factor of 5 (5, 10). There are three possibilities for a power of 3 (1,3,9). Therefore, there are 3 x 2 x 3 = 18 factors of 1800 that are multiples of 10.

Solution 2: The prime factorization of 1800 is given by $1800 = 2^3 \cdot 3^2 \cdot 5^2$. Therefore, the factors of 1800 all have factorizations of the form $2^i \cdot 3^j \cdot 5^k$, where $0 \le i \le 3$, $0 \le j \le 2$, $0 \le k \le 2$. In order to be a multiple of 10, both i and k must be at least 1. Therefore, there are 3 x 3 x 2 = 18 factors of 1800 that are multiples of 10.

Connection to... Finding the area of a triangle (Problem #9)

There are several different methods for finding the area of a triangle. $A = \frac{1}{2}bh$, where b = length of the base and h = the height of the triangle is one formula. Additionally, $A = \frac{1}{2}ab(\sin C)$ where a,b = length of two consecutive sides and C = measure of the angle included between the sides is another formula.

The formula for the area of an equilateral triangle where s = the length of one side is $A = \frac{s^2\sqrt{3}}{4}$. Heron's Formula: $A = \sqrt{s(s-a)(s-b)(s-c)}$ where s = the semi-perimeter (half the perimeter) and a, b and c are the length of the sides, is used when only the lengths of the sides are known. Research how each of these formulas for the area of triangles was derived.

Investigation & Exploration (Problem #2)

Consider the following:

a. If the crabber is unemployed the remaining months of the year, what would the mean monthly income be if figured on a 12-month basis?

b. How might you find statistics on income for fishermen to determine how typical this is?

c. How many hours per day did a crabber likely work during the 8-month season?

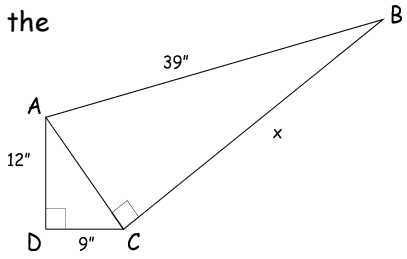
d. How many days per week did a crabber likely work during the 8-month season?

e. Using your answers from c and d, estimate the hourly wage for a crabber.

WARM-UP 5

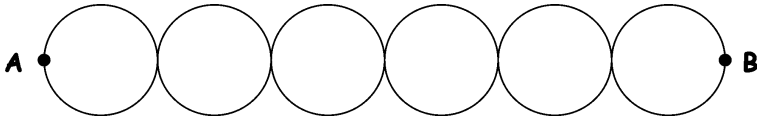
1. _____ The units digit of n^3 is 3, and n is an integer. What is the units digit of n?

2. _____ What is the number of inches in the perimeter of quadrilateral ABCD?



3. _____ A square has a side length of x units. The square's length is then increased by 2 units and its width is increased by 9 units. By how many square units does the area of the new rectangle exceed the area of the square? Express your answer in terms of x.

4. _____ The diagram shows six congruent circles with collinear centers on the x-axis. How many paths of length 3π are there from A=(0,0) to B=(6,0) if the paths must remain on the circumferences of the circles?

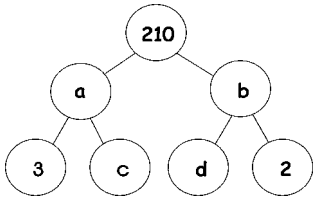


5. _____ The Los Angeles Unified School District predicts that enrollment in its schools will increase from 711,000 in 2000 to 750,000 in 2005. If one teacher is needed for each increase of 30 students, how many more teachers will be needed?

6. _____ Aimee tosses one fair 6-sided die labeled 1 through 6 and one fair 4-sided die labeled 1 through 4. What is the probability that the sum Aimee rolls is less than 5? Express your answer as common fraction.

7. _____ An arithmetic series is called a *concatenation* series if the sum of the series is represented by the concatenation of the first and last terms. For example 17+19+21+… +85 = 1785. Find a concatenation series with 41 consecutive integers and a four-digit sum. What is the sum of the integers of this series?

8. _____ What is the largest possible value of a + b + c + d in the prime factorization tree shown?



9. _____ "A laborer was hired for a year, to be paid $80 and a suit of clothes. After he worked 7 months, he left. Therefore, he only earned 7/12 of his yearly salary. For his wages, he received the suit of clothes and $35. What was the dollar value of the suit of clothes?" (paraphrased from an 1848 text by Joseph Ray, Ray's Algebra)

10. _____ Solve $2^2 \cdot 4^2 \cdot 8^2 \cdot 16^2 \cdot \cdots \cdot 1024^2 = 2^x$ for x.

WARM-UP 5

Answers

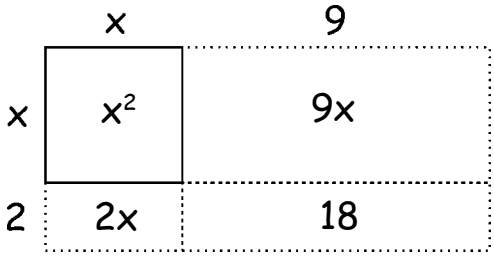
1.	7	(P, E, T)	5.	1300	(C)	8.	43	(P, E)
2.	96	(F)	6.	1/4	(T, P)	9.	28	(F, G)
3.	11x + 18	(T, M)	7.	1353	(G, E, F)	10.	110	(P, F)
4.	64	(P, C, S)						

Solution — Problem #3

A drawing may help you visualize the problem and solution.

The drawing can illustrate

- ü the original square with an area of x^2 square units
- ü a rectangle below the original square with a width of 2 and a length of x for an area of 2x square units,
- ü a rectangle on the side of the original square with a width of x and a length of 9 for an area of 9x square units
- ü a rectangle that connects the other two additions with a width of 2 and a length of 9 for an area of 18 square units.



When these areas are combined we get $x^2 + 2x + 9x + 18$ or $x^2+11x + 18$. The area could also be found by finding the product of the length and width, $(x + 2)(x + 9) = x^2+ 11x + 18$. Therefore, the new rectangle increased in area by 11x + 18 units.

****Note that this illustrates an area model for the product of two binomials.**

Connection to... History (Problem #9)

Problems from old mathematics textbooks provide glimpses into the social and economic history of the United States. What was the laborer's annual wage? Weekly wage? How long would it take your students to earn $108? Does your students' pay ever include goods or food?

One way to compare the cost of goods over time is to compare "the number of hours of work needed to purchase a particular item". In 1848, how many weeks did it take the laborer to "earn" the suit of clothes? Ask your students to estimate the cost of a "suit of clothes" now and ask them how many hours or weeks it would take for them (or their parents) to "earn" it. Check out *http://www.westegg.com/inflation/* for a calculator that will show how inflation has affected the value of a dollar over time.

Investigation & Exploration (Problem #5)

Demographics is the study of population characteristics. The results of such studies are used to make decisions regarding your community. Is the population of your community increasing, decreasing or remaining the same? Is the number of students in your school district increasing, decreasing or remaining the same? What might be contributing factors to the population growth or decline of your community?

WARM-UP 6

1. _____ The diameter of a large clock hanging in a mall in Melbourne, Australia is 3.1 meters. What is the circumference of the clock in meters? Express your answer as a decimal to the nearest tenth.

2. _____ On a particular day, 100 airplanes depart from a Babbage airport. Ten of the planes are delayed by an hour each. Of the remaining planes, half are on time and half are delayed by 20 minutes. What is the number of minutes in the average flight delay?

3. _____ Tickets numbered 1 through 100 are placed in a bag and one is randomly drawn. What is the probability that the factors of the number drawn will include 2, 3 and 5? Express your answer as a common fraction.

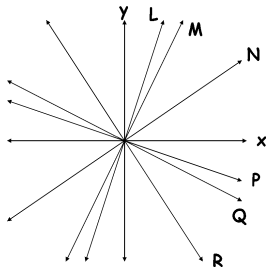
4. _____ What is the units digit of the sum: $(1!)^2 + (2!)^2 + (3!)^2 + (4!)^2 + \ldots + (10!)^2$?

5. _____ How many integers can be written as the sum of three different members of the set {2, 4, 6, 8, 10, 12, 13} ?

6. _____ A painted 2x2x2 cube is cut into 8 unit cubes. What fraction of the total surface area of the 8 small cubes is painted?

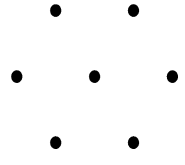
7. _____ Otto starts out facing due north. He turns to the right, first by ten degrees, then by twenty degrees, then by 30 degrees, increasing his turn by 10 degrees each time. He continues this process until he is again facing due north. How many degrees does Otto rotate in the last turn before he stops?

8. _____ Lines L, M, N, P, Q and R are drawn on the xy-plane. The slopes of the six lines (not in order) are: 2/3; 2; 3; -3/2; -1/3; -1/2. Which of the six lines (L, M, N, P, Q or R) has a slope of -1/2 ?



9. _____ The point (a, b) is reflected over the x-axis. The coordinates of the new point are (c, d). What is the value of ab + cd ?

10. _____ In the hexagonal lattice shown, each point is one unit from its nearest neighbors. In the same plane, how many circles of radius one unit pass through at least two points of the lattice?



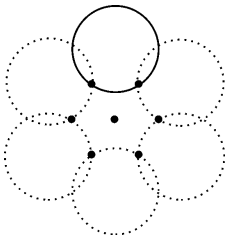
WARM-UP 6

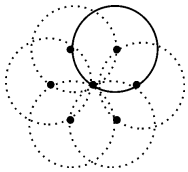
Answers

1.	9.7	(F)	5.	19	(P)	8.	Q	(M, E)
2.	15	(F, T)	6.	1/2	(M, F, P)	9.	0	(G, M)
3.	$\frac{3}{100}$	(E)	7.	80	(T, G)	10.	13	(P)
4.	7	(P, C)						

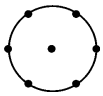
Solution — Problem #10

For this kind of counting problem, it is best to approach it with a plan of how we are going to count the objects so that we are sure to not miss any! First, let's find all of the circles of radius one unit that would have their center *outside* of the lattice points. These six circles are the only possibilities. Any circle with a center further out would either not reach any of the lattice points or would not pass through two of the points. Any circle closer in would also not pass through two lattice points. (*Notice the centers of these circles are on the perpendicular bisectors of the segments formed by each pair of consecutive outer lattice points.)





Now let's count the number of circles that have their center *on* the outer lattice points. Since the lattice points are one unit from each of their closest neighbors, then the circle would pass through the three closest neighbors of whichever outer point we picked as the center. There are six of these circles.



And finally, if the center of the circle was *inside* the outer lattice points, it could only be at the middle lattice point. The circle now goes through each of the other six lattice points. From this picture, we can see that if we shifted the center of the circle to any other interior location, the circle would not pass through two lattice points.

There are 13 circles of radius one unit, in the same plane, that pass through at least two points of the lattice.

Connection to... Geometric Representations (Problem #8)

Slopes provide a visual, geometric method for ordering fractions. To compare 2/3 and 4/5, lines may be drawn with these slopes. Looking at the steepness of the lines helps to put the fractions in order. However, what do you notice about the steepness of a line with a slope of -3/4 compared to the steepness of a line with a slope of 3/4? Graph two lines with these slopes. If you were a skier, which line would you rather ski down? Which one is steeper? How can you tell if a line has a negative or positive slope when looking at a graph?

Investigation & Exploration (Problem #6)

Answer the same question for a 3 by 3 by 3 cube. Drawing a picture may help, but can you then figure out a formula for answering the problem if we change it to a 4 by 4 by 4 cube? What happens when you use an n by n by n cube? Does the ratio of the unit cubes' painted surface area to total surface area increase, decrease or remain the same? Could you write a paragraph to a classmate explaining why this happens?

WARM-UP 7

1. _____ During the 1995 season, Oregon crabbers had a 15 million pound Dungeoness Crab harvest which sold for $24.7 million. That record was broken during the 2000 season when 15,616,728 pounds of the crabs were harvested and sold for $31,297,583. How many more cents per pound did crabs sell for in the 2000 season than the 1995 season? Express your answer to the nearest whole number.

2. _____ How many of the first 80 positive integers can be written as the sum of two distinct powers of 3? For example, $28=27+1=3^3+3^0$ can be so expressed, but 29 cannot be.

3. _____ A 3 inch by 4 inch rectangle is rotated about a corner. What is the maximum number of square inches in the area of the region touched by some point of the rectangle as it makes a full rotation? Express your answer in terms of π.

4. _____ The length, width and height of a rectangular box are each decreased by 50%. By what percent, to the nearest tenth, is the volume of the box decreased?

5. _____ A raffle was held and 1200 tickets were sold for $2.50 each. There were 17 winners. The first-prize winner received $1000. The four second-prize winners each received $250. The remaining winners each received $50. What percent of the total ticket sales was profit? Express your answer to the nearest whole number.

6. _____ Suppose that Keith's average score on four English tests is 85. The average of his three highest scores is 88.5 and the average of his three lowest scores is 82.5. What is the average of his highest and lowest test scores? Express your answer as a decimal to the nearest tenth.

7. _____ Let a*b = the least common multiple of a and b. What is the sum of all natural number values of x such that 15*x = 45 ?

8. _____ Let p and q be different prime numbers. How many positive factors will $(p^2q^4)^3$ have?

9. _____ On a trick 6-sided die the probability of rolling a 1 or 2 is each 1/4, the probability of rolling a 3 or 4 is each 1/6 and the probability of rolling a 5 or 6 is each 1/12. The trick die and a standard die are rolled. What is the probability of rolling a sum of 7? Express your answer as a common fraction.

10. _____ Choose a number. Triple the number. Add 200. Double the result. Subtract 100. Divide by 4. Subtract 150% of the original number. What is the value of the result?

WARM-UP 7

Answers

1.	36	(C)	5.	13	(C)	8.	91	(T, P, F)
2.	6	(P, E, T)	6.	83.5	(P, T, C)	9.	1/6	(T)
3.	25π	(F, M)	7.	54	(E, T)	10.	75	(F, G)
4.	87.5	(S, F)						

Solution — Problem #9

One way to solve this problem is to consider each situation on the trick die combined with the needed value on the regular die.

Trick die	Regular die	
1	6	(1/4 x 1/6) = 1/24
2	5	(1/4 x 1/6) = 1/24
3	4	(1/6 x 1/6) = 1/36
4	3	(1/6 x 1/6) = 1/36
5	2	(1/12 x 1/6) = 1/72
6	1	(1/12 x 1/6) = 1/72
		Total 1/6

A more efficient solution is to recognize that no matter what value is rolled on the trick die there is a 1/6 chance that the correct value to make a sum of 7 will be rolled on the regular die.

Solution — Problem #10

1. Call the original number N.
2. Tripling gives us 3N.
3. Adding 200 yields 3N + 200.
4. Doubling produces 6N + 400.
5. Subtracting 100 leaves 6N + 300.
6. Dividing by 4 yields $(3/2)$ N + 75.
7. Subtracting 150% of the original number N gives us $(3/2)N + 75 - 150\%N = (3/2)N + 75 - (3/2)N = 75$

Therefore, the answer is 75. Notice how the answer is not dependent on your original value of N. This type of problem is often a part of "trick" mathematical problems. Can you design a similar type of problem where the final answer is 2002?

Investigation & Exploration (Problem #2)

What if we change the question to ask how many of the first 250 integers can be written as a sum of two distinct powers of 3? Can you find a pattern or formula that may help you to find the answer if we change the question to cover the first 8000 positive integers? Change the question to ask how many of the first 8000 positive integers can be written as a sum of *three* distinct powers of 3? How does your knowledge of combinations help you?

WARM-UP 8

(For #1 and #2) The following data is reported by a company, Wheat Montana Farms: one bushel of wheat contains 1 million individual kernels which yield 42 pounds of white flour, from which 73 loaves of bread are made. One loaf of bread yields 16 slices from which 8 sandwiches are made.

1. _____ How many sandwiches can be made from one bushel of wheat?

2. _____ A Wheat Montana combine can harvest 1000 bushels of wheat per hour, on average. The amount harvested per minute, on average, could produce how many loaves of bread? Express your answer to the nearest hundred.

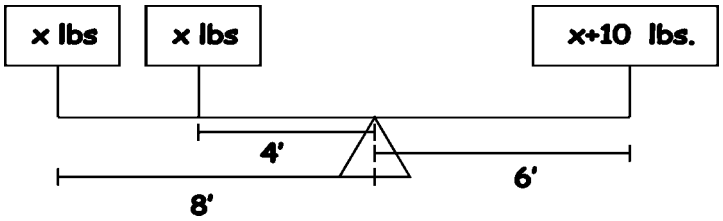
3. _____ Ray reads 18 pages the first day, 23 pages the second day, 28 pages the third day, 33 pages the fourth day and continues to add 5 more pages each successive day. How many pages will Ray read in the first fourteen days of his reading program?

4. _____ A bag contains numbered tags 1, 2, 3, 4, . . . , 200. One tag is selected at random. What is the probability that the number on the tag selected is a multiple of 3 or 7? Express your answer as a common fraction.

5. _____ Two sides of an isosceles triangle have measures of $x + 10$ and $x + 40$ and the perimeter of the triangle is 420 units. Find the sum of the two possible values of x.

6. _____ What is the units digit of $6^{10} \cdot 5^{12}$?

7. _____ What value of x, in pounds, will make the lever be in balance at the fulcrum?



8. _____ Three adults noticed a large tree along The Trail of Cedars in Glacier National Park . Together they were able to stretch their arms to form a ring around the tree with their finger tips just meeting. Their arm spans were 6 feet 1 inch, 5 feet 8 inches and 5 feet 5 inches. What is the number of inches in the diameter of this cedar? Express your answer to the nearest whole number.

9. _____ A sandbox in the shape of an equilateral triangle is 10 meters on each side. A fence is built around the triangular sandbox at a constant distance of 4 meters. What is the number of square meters in the area between the fence and the triangular sandbox? Express your answer in terms of π.

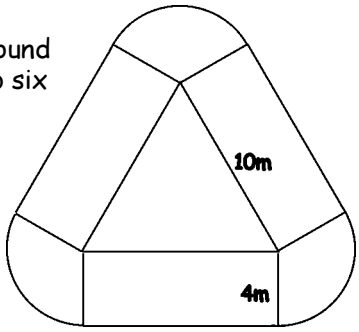
10. _____ A regular hexagon is inscribed in a unit circle. What is the number of square units in the area of the hexagon? Express your answer as a fraction in simplest radical form.

WARM-UP 8

Answers

1.	584	(C)	5.	230	(F)	8.	66	(F)
2.	1200	(C)	6.	0	(P, S)	9.	$120+16\pi$	(F, M)
3.	707	(F, P)	7.	10	(F)	10.	$\frac{3\sqrt{3}}{2}$	(F, M)
4.	$\frac{17}{40}$	(P, E)						

Solution — Problem #9



The figure to the right shows our triangular sandbox and the fence around it. Notice that the area between the fence and the sandbox is divided into six pieces. The corners are portions of circles while the sides are quadrilaterals. The fence will be linear along the sides of the triangle, running parallel to them. The corners of the fence will be curved, since all points 4 meters from a vertex of the triangle will form a circle.

Now we need to determine how large the circular and quadrilateral pieces are. Notice the quadrilaterals must be rectangles. Look at the piece with the label of 4m. It must be going perpendicular, otherwise the distance would be greater than 4m. So each rectangular region measures 4m x 10m or $40m^2$. Since there are three of these rectangular regions, they total $120m^2$.

Now for the circular regions. How much of a whole circle is each section? We can figure this out by finding the central angle. Let's look at the region at the top. There are four angles with the point at the top of the triangle as their vertex point. Two of those angles are right angles (from the rectangles) and one of the angles is 60° (from the equilateral triangle). That leaves the angle for the circular region to be 120°. That is 1/3 of a circle. Since there are three of these regions, together they will form a full circle, whose radius is 4m. So the total area for the circular regions is 16π square meters.

Putting together the rectangular and circular regions, we end up with an area equal to $120 + 16\pi$ square meters.

Connection to... Weeding out unnecessary information (Problem #1)

In the introductory paragraph for problem #1, there is a lot of information given that is really not necessary. Sometimes the most difficult part of a problem is picking out the information that is truly necessary in calculating the answer. Notice that the number of kernels, pounds of flour and slices of bread are all given, but only distract you from the information you need to answer #1. All you need is that a bushel of wheat will make 73 loaves of bread and one loaf of bread will make 8 sandwiches. Whether answering comprehension questions after reading an essay or answering word problems in math class, be sure to weed out any information that is not useful, so that you are not distracted by it. Distracting the audience with unnecessary details or actions is the idea behind many riddles, tricky test questions and even magic tricks...it works on a lot of people!

Investigation & Exploration (Problem #8)

Measure the height and arm spans of the students in your class. Make a table and a scatter plot to compare the height and arm spans (let Height be the x-axis and Arm Span be the y-axis). Fit a line to your data. How well does your data match your line? How do the heights and arm spans compare in the group of people chosen? Can you come up with an equation for your line of best fit? Enter your data into a graphing calculator and see how close your equation is to your calculator's equation for the line of best fit.

WARM-UP 9

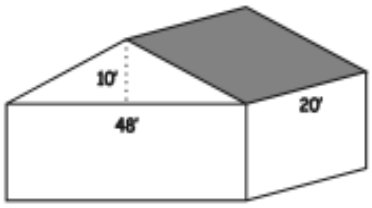
1. _____ A bag contains 30 red marbles and 50 white marbles. Twenty percent of the red marbles are removed, and each removed red marble is replaced by 4 white marbles. Next, fifty percent of the white marbles are removed, and each removed white marble is replaced by 2 red marbles. In the bag, what is the ratio of white marbles to red marbles after replacements? Express your answer as a common fraction.

2. _____ The table below shows the 1992 and 2000 salaries for employees with 10 years of experience at Seth's Surf shop. What is the positive difference between the percentage of increase in pay for a Level A employee and a Level B employee from 1992 to 2000? Express your answer to the nearest whole number.

	1992	2000
Level A employee with 10 years of experience	$36,090	$46,202
Level B employee with 10 years of experience	$17,161	$22,338

3. _____ The five-digit whole number 3a,7b1 is a perfect square. What is the greatest possible value for the product ab ?

4. _____ Adele and Toby are planning to build a new barn for their horses and want to cover the roof with tile to protect it from fire. The roof will have a slope of 5/12 known as a 5-12 pitch. Using the estimate that each tile covers a region measuring 8 x 12 inches, what is the number of tiles needed to cover just the shaded portion of the roof shown?



5. _____ A rectangle has integer side lengths and its area is equal to 24 square units. The length of each side of the rectangle is increased by one unit. What is the largest possible number of square units in the area of the new rectangle?

6. _____ The perimeter of a square is 60% of the perimeter of a triangle whose sides measure 43, 47 and 50. What is the number of square units in the area of the square?

7. _____ What is the value of the expression $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\ldots\left(1-\frac{1}{n+1}\right)$ when n = 12? Express your answer as a common fraction.

8. _____ Zan tosses one fair 6-sided die with faces labeled 1 through 6 and one fair 4-sided die with faces labeled 1 through 4. If at least one die shows a "3", what is the probability that the sum is 5? Express your answer as a common fraction.

9. _____ Jen has 15 United States coins worth 76 cents. Using any combination of pennies, nickels, dimes, quarters or half-dollars, how many different, possible combinations of coins could Jen have?

10. _____ While driving, Ron averages r miles per hour for d miles and then averages $2r$ miles per hour for d miles. Overall, Ron's average speed equals kr miles per hour. What is the value of k? Express your answer as a common fraction.

WARM-UP 9

Answers

1.	$\frac{37}{98}$	(P, M)	5.	50	(P, M)	8.	2/9	(T, P)
2.	2	(C)	6.	441	(F, M)	9.	5	(T, P)
3.	12	(P, E, G)	7.	1/13	(P, C)	10.	4/3	(S, F)
4.	780	(F)						

Solution — Problem #10

There are at least two good approaches to this problem that we recommended in the answer key above. The first method for solving the problem is to use a formula.

The equation that will get us to our answer is Average speed = $\frac{\text{Total distance}}{\text{Total time}}$

In order to find the total time, we need to add the times it took for each part of the trip. Remember that time can be found by dividing the distance by the rate of speed. So by plugging in the values from the problem we find that...

Average speed = $\frac{\text{Total distance}}{\text{Total time}} = \frac{d+d}{\frac{d}{r}+\frac{d}{2r}} = \frac{2d}{\frac{3d}{2r}} = \frac{4}{3}r$. Therefore $k = 4/3$

The next method of solving uses the idea of creating a simpler case. Since the problem must work for any choice of d and r, select "nice" numbers. For example, let's let d = 40 miles and r = 20 miles per hour. We still need to know how an average speed is found, but we won't need to work with all of the variables.

Average speed = $\frac{\text{Total distance}}{\text{Total time}} = \frac{40+40}{1+2} = \frac{80}{3}$

Therefore, $\frac{80}{3} = kr;\ \frac{80}{3} = 20k;\ k = \frac{4}{3}$.

Connection to... Interpreting Data (Problem #2)

People often use data when negotiating contracts. A Level B employee might use the data to argue that the pay for Level B employees had increased slightly more than $5,000 in 8 years while the pay for Level A employees had increased more than twice that, or more than $10,000. A Level A employee might argue that the increase in Level B employees' pay had been 30 percent while the increase in Level A employees' pay had been 28 percent over the last 8 years. It has been said that you can make data say whatever you want it to say. Look in some of your recent, local newspapers for examples of charts or graphs that may be misrepresenting data.

Investigation & Exploration (Problem #4)

The building industry uses a lot of mathematics while planning construction projects. The slope or pitch of a roof is the tangent ratio. The tangent is the ratio of the vertical change to the horizontal change in a right triangle. Builders also need to consider the slope or pitches of other things they construct. Can you think of some other structures that have a slope? Consider preparing land for a driveway to a house. In your neighborhood, take some measurements of some of the driveways and determine their slope. What about staircases or wheelchair ramps? Determine the pitch of these in your school. Are they all the same? What conditions might be considered in determining the pitch of a staircase?

WARM-UP 10

1. _____ A gallon of milk weighs 8.6 pounds. It takes 10 pounds of milk to make 1 pound of cheddar cheese. How many pounds of cheddar cheese can be made from 100 gallons of milk?

2. _____ The length of two sides of a triangle are 4 units and 7 units. The length of the third side is a whole number of units. What is the maximum possible length of the third side?

3. _____ Solve for a in terms of b if two more than twice a is three less than the square of the number which is one less than b.

4. _____ The variables c, d, e and f represent distinct digits in this correctly worked multiplication problem. What is the value of c + d + e + f ?

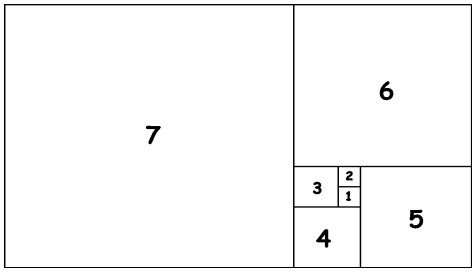
$$\begin{array}{r} 3\ c \\ \times\quad d\ 5 \\ \hline 1\ 9\ 5 \\ 1\ e\ 6\ \ \\ \hline 1\ f\ 5\ 5 \end{array}$$

5. _____ Becky has ten brown socks and ten black socks. If she randomly selects two socks from the drawer simultaneously, what is the probability that they are the same color? Express your answer as a common fraction.

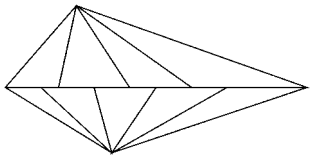
6. _____ How many ordered pairs (a,b) of positive integers satisfy the equation $3a + 4b < 12$?

7. _____ A regular hexagon ABCDEF satisfies AB=1 unit. How many square units are in the area of quadrilateral ABCE? Express your answer in simplest radical form.

8. _____ The figure is formed by beginning with a square with side length 1 (labeled with a 1) and attaching a congruent square (2). Then, at each successive step, a square is attached to the longer side of the rectangle, making a new rectangle. What is the number of square units in the area of square 7?



9. _____ How many triangles are in the figure shown?



10. _____ Suppose that x is chosen from the set {1, 2, 4, 8} and that y is chosen from the set {3, 6, 12, 24}. How many different values could x/y be?

WARM-UP 10

Answers

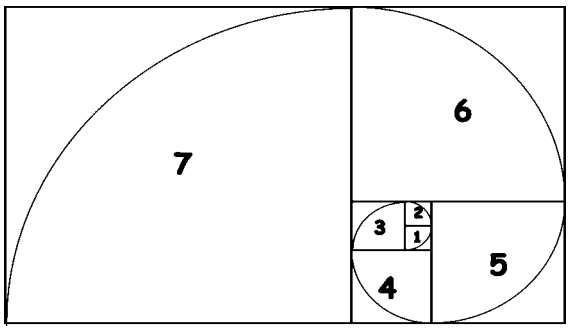
1.	86	(C)	4.	25	(E, G)	8.	169	(P)
2.	10	(F, M)	5.	9/19	(M)	9.	25	(P)
3.	$a = \frac{b^2 - 2b - 4}{2}$	(C)	6.	3	(E, G, T)	10.	7	(E, P, T)
			7.	$\sqrt{3}$	(F, M)			

Solution — Problem #10

One way to solve this problem is to think in terms of the factorizations of the integers in each set. Notice that the first set is $\{2^0, 2^1, 2^2, 2^3\}$, and the second set is $\{3\cdot2^0, 3\cdot2^1, 3\cdot2^2, 3\cdot2^3\}$. So, every fraction will be of the form $2^a/(3\cdot2^b) = 2^{a-b}/3$. The value of a-b is 3, 2, 1, 0, -1, -2, or -3, so there are 7 possible values of the fraction x/y.

Connection to... the Golden Ratio (Problem #8)

Notice how the side lengths of the successive squares form the Fibonacci sequence (1,1,2,3,5...). The spiral formed by putting a quarter circle in each square closely approximates the spiral formed by many natural forms, like seashells. For each new rectangle, the ratio of the height to the width approaches the so-called "Golden Ratio" (approximately 1.618034) which is widely seen both in art and nature. Rectangles such as these are known as golden rectangles and were believed to be the most aesthetically pleasing of all rectangles by the Ancient Greeks. Because of this, the golden rectangle can be found in many examples of their architecture.



Investigation & Exploration (Problem #3)

If Max worked out this problem and came up with $\frac{(b-1)^2-5}{2}$ as the answer and Carol worked the problem and ended up with $\frac{b^2}{2} - b - 2$ as the answer, are Max and Carol wrong? Can you show how these two answers might be shown to be equivalent to the answer provided? Would any of the answers be "more correct" than the others? Discuss the advantages and disadvantages of each form of the answer.

WARM-UP 11

1. _____ A square has an area of 144 cm^2. How many centimeters is each diagonal? Express your answer in simplest radical form.

2. _____ When buying shirts at Sport's Shirts, there is a fixed set-up fee and a constant cost per shirt. The price for 20 baseball shirts would be $390. An order for 80 baseball shirts would cost $1110. How many dollars would 140 baseball shirts cost at Sport's?

3. _____ From an 1855 text, School Arithmetic, by Charles Davies: "If twenty grains make one scruple; three scruples make one dram; eight drams make one ounce; and twelve ounces make one pound; what part of an ounce is 3/10 of a scruple?" Express your answer as a common fraction.

4. _____ The number N is a positive multiple of both 6 and 8. N is also a factor of 432. How many different integers can N be?

5. _____ Sara is painting the four walls of her room. Her room is 8 feet wide by 10 feet long, and all of the walls are the same height. One can of paint covered exactly half of one of the smaller walls. How many more cans of paint, of the same size, will she need to paint the rest of the room?

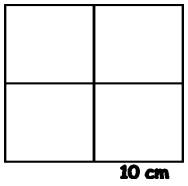
6. _____ The quantities x and y vary inversely and x = 27 when y = 9. Determine the value of x when y = 60. Express your answer as a mixed number.

7. _____ A news article reported the length, in months, of investigations of jet accidents involving commercial flights of U.S. carriers originating in the United States from 1964-1996. The data is provided in the stem-and-leaf plot. What is the positive difference, in months, between the median and the mode? Express your answer as a decimal to the nearest tenth.

Length, in months, of Investigations

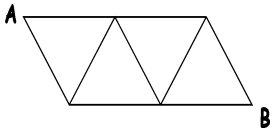
0	5 6 6 6 7 7 7 7 8 8 8 8 8 8 9 9 9
1	0 0 0 0 0 1 1 2 2 2 3 3 3 4 4 4 4 4 5 5 5 5 5 6 7 7 8 9
2	1 1 3 7 8
3	0 4
4	9
5	4

8. _____ A coin of diameter 2 cm is dropped randomly on the tabletop shown so that the entire coin lies on the tabletop. Each square tile is 10 cm on a side. What is the probability that the coin lies completely within one of the squares? Express your answer as a common fraction.



9. _____ Use each of the digits 2,3,4,6,7 and 8 exactly once to construct two three-digit integers M and N so that M-N is positive and is as small as possible. Compute M-N.

10. _____ In this diagram, each short line segment has length 1. The shortest paths from A to B each have length 3. How many paths of length 4 are there from A to B? Assume that you can change direction only at the four vertices.



WARM-UP 11

Answers

1.	$12\sqrt{2}$	(F, M)	5.	8	(F, M)	8.	$\frac{64}{81}$	(M, F)
2.	1830	(F, G)	6.	$4\frac{1}{20}$	(C, F)	9.	39	(G, P)
3.	1/80	(C, M)	7.	4.5	(C)	10.	6	(M, P)
4.	6	(P, G)						

Solution — Problem #9

If we are trying to make M-N as small as possible, then we are trying to make the two numbers as close to each other as possible. We can arrange for M-N to be less than 100 by choosing the hundreds digit for M to be one more than the hundreds digit for N . There are several ways to do this. You also want to keep the tens digit of the greater number as little as possible, while keeping the tens digit of the lesser number as large as possible. Using the 2 and 8 for the tens digits still leaves the 6 and 7 for the hundreds digit. You may also try some trial and error and see that this also reveals that M=723 and N=684 are as close to one another as possible. Thus the answer we seek is 723-684 = 39.

Connection to... Science (Problem #6)

Direct variations and inverse variations are used to explain the relationship between the volume of a gas and temperature or pressure. The volume of a gas increases as the temperature increases while the volume of a gas decreases as the temperature decreases, and the quantities are directly proportional. The letter k represents the constant of proportionality. This is often represented algebraically as $V = kT$ or $V_1/V_2 = T_1/T_2$.

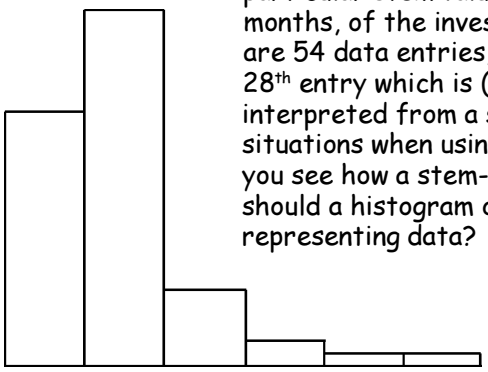
On the other hand, the volume of a gas decreases as the pressure exerted on the gas increases while the volume of a gas increases as the pressure decreases. This illustrates an inverse variation and is often represented as $V = k/p$ or $V_1/V_2 = P_2/P_1$.

Check a chemistry book to determine the names of these laws or a law that relates the temperature of a gas with the pressure exerted on the gas.

Can you think of other real world settings where direct or inverse variations model the relationship between two quantities?

Investigation & Exploration (Problem #7)

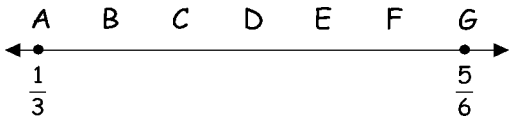
A stem-and-leaf plot is a quick way to picture the shape of a distribution while including the actual numerical values in the graph. The stem of the plot is a vertical number line that represents a range of data values in a specified interval. The leaves are the numbers that are attached to the particular stem values. In the stem-and-leaf plot for #7, the length, in months, of the investigations ranged from 5 to 54. The mode is 8. There are 54 data entries, an even number. The median is the mean of the 27th and 28th entry which is (12+13)÷2 = 12.5. This is a lot of information that can be interpreted from a single representation of the data. What are some situations when using a stem-and-leaf plot is the best representation? Can you see how a stem-and-leaf plot is easily turned into a histogram? When should a histogram or pie chart be used? What are some other ways of representing data?



WARM-UP 12

1. _____ Ernie has 450 Galaxy cards. Sara has 10 more than Ernie and twice as many as Bert. How many Galaxy cards do they have altogether?

2. _____ The points A through G are evenly spaced on the number line. If A = 1/3 and G = 5/6 what is the value at point C, expressed as a common fraction?



3. _____ The number y is 125% of another number x. What percent of 8y is 5x?

4. _____ The five-digit number 4a,ab7 is divisible by nine where a and b are single digit whole numbers. How many possible combinations are there for a and b?

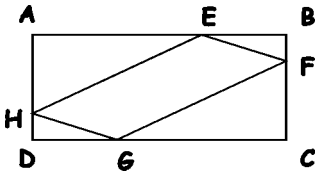
5. _____ How many integers in the range 500 to 999 have no consecutive identical digits? For example, 626 is an integer with no consecutive identical digits, but 722 is not.

6. _____ In a coordinate plane, point A (4, -2), is reflected over the x-axis and labeled A'. A' is reflected over the y-axis and labeled A". What is the sum of the coordinates of point A"?

7. _____ A rectangle has dimensions 20 inches by 30 inches. If one side is increased by 30% and the other side is decreased by 20%, then what is the largest possible number of square inches in the area of the new rectangle?

8. _____ Three friends, Ralph, Emerson and Waldo, each select a number from the set {1,2,3,11,12,13,21,22,23} and remove it. Then they add their three numbers together. If they put their numbers back and repeat this process for all possible combinations, how many different sums can they get?

9. _____ ABCD is a rectangle, AB = 6, BC = 4, EFGH is a parallelogram, AE/BE=2/1, and BF/FC = 1/3. What is the ratio of the area of parallelogram EFGH to the area of rectangle ABCD? Express your answer as a common fraction.



10. _____ The sequence 2,3,5,6,7,8,10,11 … consists of the positive integers which are not perfect squares. What is the 100th member of the sequence?

WARM-UP 12

Answers

1.	1140	(C)	5.	405	(P)	8.	29	(P, T)
2.	1/2	(M, F)	6.	-2	(M)	9.	$\frac{5}{12}$	(F)
3.	50	(F, G)	7.	624	(F)	10.	110	(P, E)
4.	11	(P, T)						

Solution — Problem #3

Direct translation of words into math symbols is a good approach to this problem. With this kind of problem, we can remember that the word "is" translates into "=" and the word "of" translates into multiplication.

I. The number y is 125% of another number x. $y = 1.25(x)$

II. What percent of 8y is 5x? $? (8y) = 5x$

We will be able to manipulate the second sentence by using the first sentence. From sentence I, we know that we can substitute 1.25x for y when we work with sentence II. Therefore sentence II becomes: $? (8(1.25x) = 5x$. This can be simplified to: $? (10x) = 5x$. And this will bring us to our final answer... $? = (5x)/(10x) = 1/2 = 50\%$.

Solution — Problem #4

A number is divisible by nine if the sum of its digits is divisible by nine. The sum of the digits of 4a,ab7 is 4 + a + a + b + 7 = 2a + b + 11. The expression 2a + b + 11 must be equal to 18, 27 or 36 in order for 2a + b + 11 to be divisible by 9.

(Why can't 2a + b + 11 be equal to 0, 9, 45, 54, 63, . . .?)

	2a + b + 11 = 18		2a + b + 11 = 27		2a + b + 11 = 36
	2a + b = 7		2a + b = 16		2a + b = 25
If	a = 0, b = 7	*If*	a = 4, b = 8	*If*	a = 8, b = 9
	a = 1, b = 5		a = 5, b = 6		a = 9, b = 7
	a = 2, b = 3		a = 6, b = 4		
	a = 3, b = 1		a = 7, b = 2		
			a = 8, b = 0		

There are 11 combinations of a and b that will make 4a,ab7 divisible by 9.

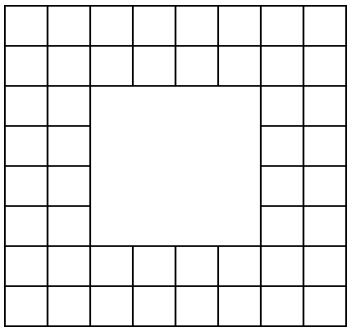
Investigation & Exploration (Problem #4)

If graphed on the Cartesian coordinate grid, where x = a and y = b, what will the scatter plot of the 11 combinations look like? Investigate the connection between the graph and the solution. Can you use what you just discovered to find all of the possible values for c and d to make 1d,3cc,dc4 divisible by 9? Though the graphing method may not be the quickest way to solve this problem, it brings up many interesting questions and connections. How is this similar to or different from the original problem in the Warm-Up?

WARM-UP 13

1. _____ In March 2000, demolition experts used 4461 pounds of gelatin dynamite to bring down the Kingdome, a stadium in Seattle. The dynamite was stuffed into 5905 holes throughout the dome. What was the mean number of ounces of dynamite per hole? Round your answer to the nearest whole number.

2. _____ How many squares are determined by the grid lines to the right if the 48 smaller quadrilaterals are congruent squares?



3. _____ The population of the United States is 275 million people. The land area of the United States is 3.6 million square miles. If the land area of the United States were equally divided among all of its population, each person would 'own' a square piece of land K feet by K feet. Calculate K to the nearest foot.

4. _____ Two years ago the ages of an oak tree and that of a younger oak tree were both perfect squares. Two years from now both of their ages will be perfect cubes. What is the sum, in years, of the current ages of the two oak trees?

5. _____ Miami-Dade County, Florida school officials reported 84,000 students attending classes in portable classrooms in the fall of 2000. If the mean number of students per portable classroom is 30, how many portable classrooms were being used?

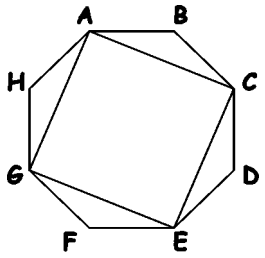
6. _____ The product of the digits of a four-digit number is 6!. What is the smallest possible value of this number?

7. _____ Two standard 6-sided dice are tossed. What is the probability that the sum of the two numbers rolled is greater than nine? Express your answer as a common fraction.

8. _____ Al, Ed and Tom are different ages. Exactly one of the following three statements is true: I. *Ed is the oldest.* II. *Al is not the oldest.* III. *Tom is not the youngest.* Who is the youngest boy?

9. _____ What is the sum of the coordinates of the point on the line $x + y = 4$ closest to the origin?

10. _____ Square ACEG is inscribed in the regular octagon ABCDEFGH as shown. What is the ratio of the area of square ACEG to the area of octagon ABCDEFGH? Express your answer as a fraction in simplest radical form.

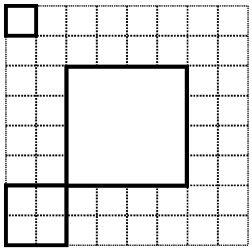


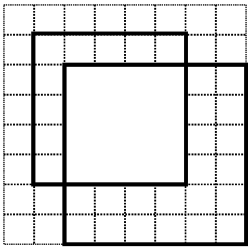
WARM-UP 13

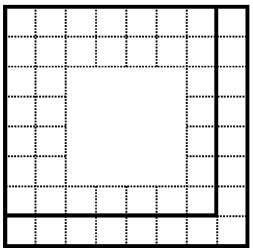
Answers

1.	12	(F, C)	5.	2800	(C)	8.	Ed	(T)
2.	91	(P, S)	6.	2589	(P, G, E)	9.	4	(M, G)
3.	604	(C)	7.	1/6	(P, T)	10.	$\frac{\sqrt{2}}{2}$	(F)
4.	129	(G, T)						

Solution — Problem #2







For this kind of counting problem, it's best to have a plan for how you will count each square. Probably the easiest way will be to count the number of squares of a given size. We'll start with the smallest, a 1x1 square:

1x1 squares (shown in the first diagram)	48
2x2 squares (first diagram)	24
3x3 squares	0
4x4 squares (first diagram)	1
5x5 squares (shown in second diagram)	4
6x6 squares (second diagram)	9
7x7 squares (shown in third diagram)	4
8x8 squares (third diagram)	1

A grand total of 91 squares!

Solution — Problem #8

Analyze each of these three possibilities (A,B and C):

	A	**B**	**C**
Statement I	T	F	F
Statement II	F	T	F
Statement III	F	F	T

Possibility A: Since II, "*Al is not the oldest,*" is false, we know that Al is the oldest. But this contradicts the true statement, I, "*Ed is the oldest.*" Possibility A is impossible.

Possibility B: Since III, "*Tom is not the youngest,*" is false, Tom is the youngest. However, since "*Ed is the oldest* " is false, while "*Al is not the oldest* " is true, no one can be the oldest. Possibility B is impossible.

Possibility C: Since II, "*Al is not the oldest,*" is false, Al is the oldest. Since III, " *Tom is not the youngest,*" is true, only **Ed** can be the youngest. Ed being the youngest is allowed since I, "*Ed is the oldest,*" is false.

Connection to... Social Studies (Problem #3)

The population density of countries varies dramatically and can be represented in many ways. The image of a 609 foot by 609 foot square for each United States resident is quite different from a 1894 foot by 1894 foot square for each Canadian or a 43 foot by 43 foot square for each resident of Singapore.

At the website http://www.undp.org/popin/wdtrends/6billion/t09.htm, the 1999 population densities for all countries is given in the unit "people per square kilometer." For example, the United States of America has 29 people per square kilometer. A challenging arithmetic problem is to convert between "people per square kilometer" and "N foot by N foot square per person." Can you develop a formula for this conversion?

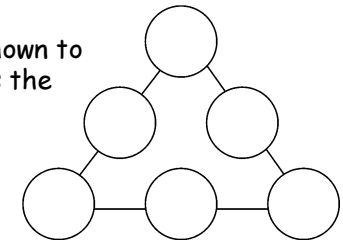
WARM-UP 14

1. _____ The point (2,3) is reflected over the x-axis to point P. Then P is reflected over the line y=x to point Q. What is the x-coordinate of Q?

2. _____ What is the largest value of n for which 2^n divides 10! ?

3. _____ It was reported that California's non-Hispanic white population was 57 percent of the total California population in 1990 and 49.8 percent in 1999. There were 17 million non-Hispanic whites in California in 1990 and 16.5 million in 1999. By how many people did California's population increase from 1990 to 1999? Express your answer to the nearest million.

4. _____ The numbers 1,2,3,4,5 and 6 are placed in the six circles shown to the right so that the sum along each edge of the triangle is the same. What is the smallest possible sum along an edge?



5. _____ Given six distinct points on a line, how many distinct segments can be named using the six points?

6. _____ What is the area, in square units, of the largest regular hexagon which can be inscribed in an equilateral triangle with area 24 square units?

7. _____ A customer purchased a large circular pizza with a 12-inch diameter. When she began eating the pizza at home, she found an unusual amount of crust on the outer edges with no sauce, cheese or toppings. She measured the outer edge in a number of places and found it to be a 2-inch border all the way around the pizza. What percent of the pizza was without sauce, cheese or toppings? Round your answer to the nearest whole number.

8. _____ What is the sum of all prime numbers between 35 and 70 which, when divided by 12, leave a remainder that is a prime number?

9. _____ A point is randomly selected from within the rectangle having vertices at (0,0), (2,0), (2,3) and (0,3). What is the probability that the x-coordinate of the point is less than the y-coordinate? Express your answer as a common fraction.

10. _____ A circle has a radius of 50 meters. The radius is increased by 40%. By what percent is the area increased?

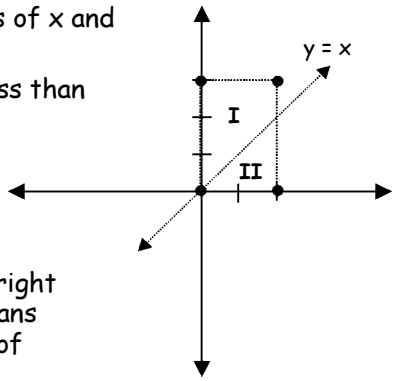
WARM-UP 14

Answers

1.	-3	(M)	5.	15	(F, M)	8.	310	(E, G, T)
2.	8	(P, E, S)	6.	16	(F, M)	9.	2/3	(F, M)
3.	3,000,000	(F, C)	7.	56	(F, M)	10.	96	(F, C, M)
4.	9	(G)						

Solution — Problem #9

Graphing the information in this problem will help us to visualize what we are looking for. It will probably help us to avoid a common mistake as well...it's important to remember that we are not only considering integer values of x and y!



If we are trying to find the points where the x-coordinates are less than the y-coordinates, we are working with the points in region I.

The problem states we are able to choose any point from the rectangle and then must see what the probability is that it will be inside region I. We can figure this out by using the area of the regions. The rectangle is 2x3 and has an area of 6 square units. The area of region II may be easier to find than region I since it is a right triangle with legs of 2 units each. Its area is 2 square units. That means that the area of region I is 6-2 or 4 square units. So the probability of being inside region I when already inside the rectangle is 4/6, or 2/3.

Connection to... Number games (Problem #4)

How many different ways can you arrange these integers to get a sum of 9? Some triangles have the numbers in the same relative positions, but are obtained by rotating or reflecting the triangle. How many of these are there? Another way to get new arrangements is what we call the "dual" where every number is moved to the opposite side of the triangle, so numbers on the edges become numbers on the corners, and vice versa.

There are many number games where numbers are arranged in patterns according to their sums or products. Most students are familiar with Magic Squares. There are many publications devoted to math puzzles. These puzzles help with critical thinking, problem solving and general number sense...they also make math practice fun! Students can develop their own number crosswords, with math problems as clues, and swap them with a buddy.

Investigation & Exploration (Problem #1)

A Reflection is a type of transformation used in math. Not only can you reflect one point through a line of reflection, but you can reflect entire shapes as well. Graph three points and connect them so that they form a triangle. If you reflect those three points over the x-axis, what do you notice about the coordinates of the new points compared to the coordinates of the original points? Take this new triangle and now reflect the points across the line y=x. Again, notice how the coordinates change. Does the size or shape of the triangle change? Where do you think a point reflects that is located on the line of reflection? Given the pattern of the changes that occurred with the coordinates of the three vertices of the triangle, can you determine, without graphing it, where the point (-3, 5) would end up if you were to reflect it over the x-axis and then over the line y=x? Write instructions to a friend telling her how she can determine, without graphing, where the point (6, -3) will end up if she reflects it over the line y=x and then over the y-axis.

WARM-UP 15

1. _____ Two lines defined by the equations y=mx+4 and y=3x+b, where m and b are constants, intersect at the point (6,10). What is the value of b+m?

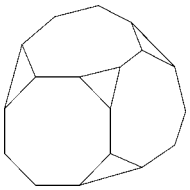
2. _____ If the greatest common divisor of a and b is 100, what is the greatest common divisor of $3a^2$ and $3b^2$?

3. _____ Jabbar wants to choose a height for his son's basketball hoop that is in the same proportion to his son's height as the standard 10 foot hoop is to the average 6 foot 6 inch professional basketball player. His son is 4 feet 4 inches tall. What is the number of inches in the height at which he should place the basket?

4. _____ P and Q are positive prime numbers with P > Q and P + Q = 124. What is the smallest possible value of P – Q?

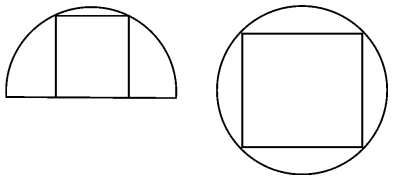
5. _____ What is the least natural number greater than 7 that has a remainder of 7 when divided by 24 and also has a remainder of 7 when divided by 32?

6. _____ The eight corners of a cube are snipped off to form a polyhedron with 6 octagonal and 8 triangular faces. What is the fewest number of colors that can be used to paint the faces so that no pair of adjacent faces are the same color?



7. _____ A store offers customers a card with 5 circles, each hiding a percent of discount: 50%, 50%, 25%, 10%, 5%. The customer selects two circles to uncover and receives a discount equal to the average of the two values. What is the probability of receiving a 50% discount? Express your answer as a common fraction.

8. _____ What is the ratio of the area of a square inscribed in a semicircle with radius 10 inches to the area of a square inscribed in a circle with radius 10 inches? Express your answer as a common fraction.



9. _____ There are 67 people in a tennis tournament. A player is eliminated from the tournament after losing 4 matches. What is the maximum number of matches that can be played so that exactly 5 people are left in the tournament?

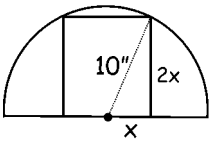
10. _____ A circle, a square and a triangle are all drawn in a plane. None of the square's sides are collinear with the sides of the triangle. What is the largest number of points that can belong to at least two of the three figures?

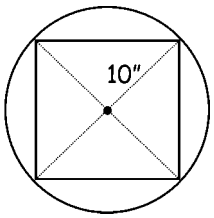
WARM-UP 15

Answers

1.	-7	(M, C, F)	5.	103	(G, E)	8.	2/5	(F, C)
2.	30,000	(P, S)	6.	4	(M, G)	9.	263	(T, P)
3.	80	(C, F)	7.	1/10	(T)	10.	20	(M)
4.	18	(E, G)						

Solution — Problem #8





To find the area of the square inscribed in the semicircle, let x equal one-half the length of the side of the square. Then 2x equals the length of the side of the square. We can then use the Pythagorean Theorem on the right triangle we form by inserting the dotted line shown (a radius of the circle). By the Pythagorean Theorem, we have $10^2 = x^2 + (2x)^2$. Solving for x, we get $x = 2\sqrt{5}$. The area of this square is $(2x)^2 = \left(2\left(2\sqrt{5}\right)\right)^2 = 80$ square inches.

To find the area of the square inscribed in the circle, we can use the formula for finding the area of a rhombus which uses its diagonal measures, $\frac{d_1 \cdot d_2}{2}$ or $\frac{20 \cdot 20}{2} = 200$ square inches.

The problem asks for the ratio of the area of the square inscribed in the semicircle to the area of the square inscribed in the circle, so we have (80)/(200) or 2/5.

Connection to... Sports (Problem #3)

Many times accommodations are made in sports for younger athletes. Steps are taken to ensure that the challenge of the game is not too overwhelming because of a youngster's smaller size. The most common change is in the size of the playing area; smaller basketball courts, baseball fields and soccer fields are used for younger players. Other times the size of the equipment is altered or the length of playing time is shortened. In the game of tennis, however, the court size is not altered and the size of tennis balls remains the same. Suppose you would like to adapt the game of tennis for a younger player. Research the dimensions of the standard tennis court, the height of the net and the diameter of the ball. Assume these measurements are good for a 6 foot adult. What would the new measurements be if you were to change them proportionally for a 4 foot 2 inch boy? Compare the ratio of change for the height of the net, the surface area of the court and the volume of the ball. Do you see a connection?

Investigation & Exploration (Problem #6)

One of the most famous mathematical theorems proven in the 20th century was the "Four Color Theorem". This theorem establishes that *any* map in the plane can be colored with four colors, so that no adjacent regions have the same color. Of course, some maps can be colored with only three, or even two colors. How many colors are needed to color a cube so that adjacent faces do not share the same color? A tetrahedron? Try drawing a map in the plane which cannot be colored with fewer than four colors. The history of this theorem is very interesting, with many failed attempts at proving it. Not until the 1970's, with the aid of a computer program taking 1200 hours to run, did many people believe an actual proof had been found!

WARM-UP 16

1. _____ Palavi's score on a quiz was 78, which was 130% of Samidh's score. What was Samidh's score?

2. _____ When purchasing 3 hamburgers at $2.40 each at the local restaurant, the customer receives an additional hamburger free of charge. Four people evenly share the cost of four hamburgers. What is the number of dollars, to the nearest hundredth, in each person's share?

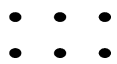
3. _____ The puzzle pictured consists of three identical cubes which may be twisted until each vertical face is a single color. What color is the face marked "?" ?

4. _____ How many different whole numbers between 0 and 1000 have digits whose sum is 9?

5. _____ Let P(n) and S(n) denote the product and the sum of the digits of the integer n, respectively. For example, P(23)=6 and S(23)=5. How many two-digit numbers N satisfy N=P(N)+S(N).

6. _____ Given the line y = 2x – 10, what is the length, in units, of the longest segment of this line that lies in Quadrant IV? Express your answer in simplest radical form.

7. _____ There are eight unit squares in the plane that have two or more vertices in this 2 by 3 array of lattice points. How many unit squares have at least two vertices in an m by n array of lattice points?



8. _____ Two points are randomly selected from the set of ordered pairs {(0,0), (1,0), (2,0), (0,1), (1,1), (2,1)}. What is the probability that they are one unit apart? Express your answer as a common fraction.

9. _____ The sum of the squares of two whole numbers is 160. The product of the numbers is 48. What is the absolute value of the difference between the two whole numbers?

10. _____ Find an integer n satisfying $\frac{(n+5)!}{n!(n+4)} = \frac{9!}{3!}$.

WARM-UP 16

Answers

1.	60	(C)	5.	9	(P, E, G)	8.	7/15	(E, G)
2.	1.80	(C)	6.	$5\sqrt{5}$	(M)	9.	8	(E, G)
3.	Red	(P, E)	7.	mn+m+n-3	(P)	10.	13	(C)
4.	55	(P, E, G)						

Solution — Problem #10

Solution 1:

Expanding and canceling yields the equation $\frac{(n+5)!}{n!(n+4)} = (n+5)(n+3)(n+2)(n+1) = 60{,}480$.

Now break 60,480 into its prime factorization: $(n+5)(n+3)(n+2)(n+1) = 2^6 \cdot 3^3 \cdot 5 \cdot 7$

By observation we see that we need to use the prime factors and their powers to form four integers that multiply to 60,480 and are almost consecutive, except that the largest will be two bigger than the next largest. Going through possible choices of combinations will not take long. Trying to use the 5 and 7, we see that the product of 5,6,7 and 9 is not large enough. If we multiply the 7 by a 2, then we must also make the 5 larger to keep them within a five integer range of each other. But if we multiply the 5 by a 2, this leaves 10, _ , _ , and 14 and there isn't any way to make an 11 out of the prime factorization integers left. If we try to multiply the 5 by a 3 instead of the 2, we end up with 15 and 14 and a good set of factors left to make 16 and 18...so we have 14,15,16 and 18, all of which do multiply to 60,480. So we now know that $n + 1 = 14$ and $n = 13$.

Solution 2:

Again, expand and cancel, which gets us $\frac{(n+5)!}{n!(n+4)} = (n+5)(n+3)(n+2)(n+1) = 60{,}480$.

Therefore, 60,480 is approximately $(n + 3)^4$. Thus, $n+3 \approx \sqrt[4]{60{,}480} \approx 15.68$. Trying $n = 13$ yields a solution.

Connection to... Games (Problem #3)

The puzzle *Instant Insanity* is played with four different cubes, each with faces that are red, white, green or blue. The puzzler is asked to stack the four cubes so that each side of the new (vertical) box shows each of the four colors. This may seem like a simple task at first, but there are over 40,000 combinations to try if you just go at it randomly. Check out the many websites dedicated to this age-old game to see how the game works and to find good ways to approach finding the solution. There are also several books about discrete mathematics that address and solve this problem.

Investigation & Exploration (Problem #5)

Are there any three-digit numbers that satisfy this property? If so, what are they? If not, can you write a convincing argument that no such numbers exist? What would be some good numbers to investigate?

WARM-UP 17

1. _____ According to a published report during the summer of 2000, the cost for long-term parking per day at 40 U.S. airports ranged from $3.00 in Little Rock, AR to $18.00 in Seattle, WA. For the year 2000, how many more total dollars in revenue are produced by a long-term parking space at the Seattle airport than a long-term parking space at the Little Rock airport, if each parking space is used every day?

2. _____ A song is written in $\frac{3}{4}$ time, meaning that there are 3 beats to the measure and each quarter note gets one beat. A half note would get 2 beats and an eighth note gets 1/2 beat. A practice exercise consists of 26 eighth notes, 26 quarter notes and 21 half notes played consecutively. What is the number of measures in the exercise?

3. _____ What is the y-intercept of the line that is the perpendicular bisector of the segment joining the points (4,7) and (-4,12)? Express your answer as a common fraction.

4. _____ Let S = # of square units in the surface area of a cube. Let V = # of cubic units in the volume of the cube. When the side length of the cube is doubled, by what factor is the ratio of S to V multiplied? Express your answer as a common fraction.

5. _____ Each term in a sequence is the sum of the previous two terms. If the sequence contains the terms a, b, c, 12, 19 and 31, in that order, what is the value of a?

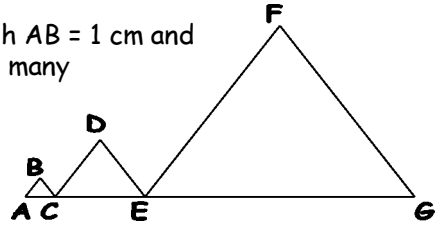
6. _____ The Riverside Bed and Breakfast Inn serves three married couples breakfast at their round table which seats exactly six people. The hostess wants to seat guests so that no husband and wife sit next to each other and the guests alternate male and female. How many different arrangements are possible? A rotation of an arrangement is not considered a different arrangement.

7. _____ The diameters of two circles are 8 inches and 12 inches. The area of the smaller circle is what percent of the area of the larger circle? Express your answer to the nearest whole number.

8. _____ What is the least whole number n such that 84 divides n! ?

9. _____ Three faces of a cube are randomly selected. What is the probability that they have a common vertex? Express your answer as a common fraction.

10. _____ Triangles ABC, CDE and EFG are equilateral with AB = 1 cm and CD = 3 cm. Points B, D and F are collinear. How many centimeters are in the length of segment EF?

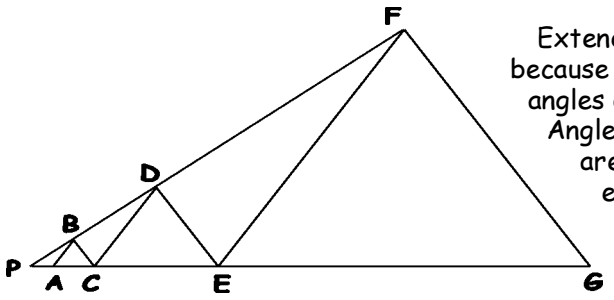


WARM-UP 17

Answers

1.	5490	(C)	5.	2	(C)	8.	7	(G, E)
2.	27	(M, C)	6.	2	(P, M)	9.	2/5	(T,M)
3.	$\frac{19}{2}$	(F)	7.	44	(F)	10.	9	(F)
4.	1/2	(F)						

Solution — Problem #10



Extend line BF to intersect line AG at point P. Now because of the angle at P and the fact that all of the angles of the triangles are 60°, we can use the Angle Angle Postulate to see that ΔPAB, ΔPCD and ΔPEF are all similar. Now we can set up the following extended proportion with the length of segment $PA = x$:

$$\frac{PA}{AB} = \frac{PC}{CD} = \frac{PE}{EF} \quad \text{or} \quad \frac{x}{1} = \frac{x+1}{3} = \frac{x+4}{n}$$

Solving the first equality, we get $3x = x + 1$ or $x = 1/2$. Plugging this into the second equality and cross-multiplying, we get $n = 9$ centimeters.

Connection to... Biology (Problem #4)

Some biologists study the ratio of the surface area to the volume of living organisms. As you saw in this problem, increasing the size of an object without changing its shape causes the surface area to volume ratio to decrease. For living creatures, the volume determines the amount of body heat produced, while the surface area determines the amount of body heat which is radiated, or lost, to the surroundings. For this reason, smaller creatures, which have greater area relative to their volume, tend to have higher metabolisms and body temperatures. For example, mice have a much higher metabolism than elephants.

Additionally, smaller animals tend to have a more compact spherical shape, while larger animals have more and longer appendages, which tends to increase their surface area to volume ratio.

Investigation & Exploration (Problem #5)

This is an example of a Fibonacci - like sequence, in which each term is the sum of the two previous terms. How far backwards can you extend the sequence? What happens if you choose other values for the final three terms? What would happen if each term is the sum of the previous three terms?

The actual Fibonacci sequence (0,1,1,2,3,5,8,...) is extremely popular and very useful in mathematics. Its various connections to nature are amazing. It is also related to Pascal's triangle, another formation of numbers that involves finding the sum of two previous numbers. Can you find where the Fibonacci sequence can be seen in Pascal's Triangle?

WARM-UP 18

1. _____ A news source reported that 37 percent of the Asian population in the United States (4 million people) live in California. What is the total Asian population in the United States? Express your answer to the nearest million.

2. _____ The sum of an integer and its square is 6 less than the square of the next greater integer. What is the value of the integer?

3. _____ Tyler rolls four standard 6-sided dice and finds that the product of the numbers rolled is 450. What is the sum of the numbers that were rolled?

4. _____ What integer is closest to the number of square units in the area of a triangle whose sides are 2, 3 and 3?

5. _____ With two games remaining in the baseball season, two players have nearly identical batting averages. McKay has 197 hits in 580 at bats. Nickels has 196 hits in 579 at bats. In the last two games McKay has 6 hits in 10 at bats and Nickels has 5 hits in 8 at bats. What is the positive difference between their final batting averages? Express your answer as a decimal to the nearest thousandth.

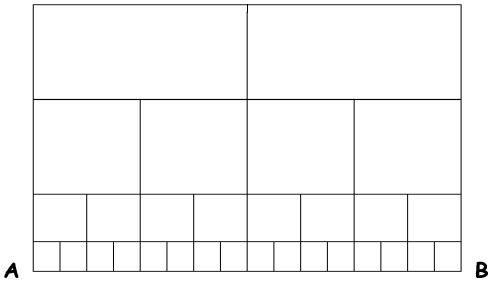
6. _____ A point is randomly selected from within the triangle having vertices at (0,0), (2,0) and (0,3). What is the probability that the point is within one unit of (0,0)? Express your answer as a common fraction in terms of π.

7. _____ A pentagon P has vertices A=(0,0), B=(7,0), C=(13,8), D=(5,14) and E=(0,14). Line L passes through the origin and divides P into two quadrilaterals with equal perimeters. What is the sum of the coordinates of the point F where L meets segment CD? Express your answer as a decimal to the nearest tenth.

8. _____ What is the sum of the digits of the decimal representation of $\frac{10^{22}+8}{9}$?

9. _____ What is the remainder when the product of the first five primes is divided by 12?

10. _____ In the grid to the right, it costs exactly \$1 to move from any vertex to an adjacent vertex. How many dollars does it cost to go from point A to point B along the cheapest path?



WARM-UP 18

Answers

1.	11,000,000	(C)
2.	5	(C)
3.	19	(C, E, G)
4.	3	(F, M)
5.	.002	(C)
6.	$\frac{\pi}{12}$	(E, M, C)
7.	19.8	(M)
8.	23	(P)
9.	6	(C)
10.	10	(P, M, G)

Solution — Problem #8

At first, this problem may look like it's going to take a lot of work and a calculator with a giant screen to show all the digits in our final number! But there's really not a lot of calculating to be done. We just have to find a better way to write the problem. Watch how we can transform the expression.

$$\frac{10^{22}+8}{9}=\frac{10^{22}-1+9}{9}=\frac{10^{22}-1}{9}+1$$

Now, looking at a pattern, we can see that $(10^1 - 1)/(9) = 1$, $(10^2 - 1)/(9) = 11$, $(10^3 - 1)/(9) = 111$,... Notice how the answer is always going to be an integer made up of an amount of 1's equal to the exponent of the 10. From this we can tell that $(10^{22} - 1)/(9)$ is going to be a 22-digit number with all digits of 1. But if we then perform the last operation of adding 1 to the number, it's still a number with 22 digits, but would look like this: 1111...1112. Its digits add to 23.

Another way to approach this problem is to realize that the original numerator of the fraction will be a 23-digit number that starts with 1, has 21 zeros and ends in 8. When we divide this number by 9, we get a 22-digit quotient with 21 ones and a final 2. The sum of these digits is $21 \cdot 1 + 2 = 23$.

Connection to... Baseball (Problem #5)

In this problem we are looking at calculating batting averages for baseball. Baseball is a game that uses many different statistics to determine how well a player is doing throughout the season. Not only is a player's batting average used to determine his effectiveness on offense, but also his on-base percentage and his slugging percentage. What are the differences between how these three numbers are calculated for a player?

The position of pitcher has its own set of statistics. The primary statistic is a pitcher's earned run average (ERA). Other numbers that are often considered when determining how well a pitcher is performing are his strike-outs per inning as well as pitches per inning. The next time you watch a televised baseball game, notice how many references are made to math-related facts!

Investigation & Exploration (Problem #9)

What is the remainder when the product of the first six primes is divided by 12? Based on your answers to this question and problem #9, develop a conjecture and prove it or find a counter-example. What if we changed the problem to ask, "What is the remainder when the product of the first six primes is divided by 9"? Can you find an exact answer or narrow it down to a limited number of possibilities?