WORKOUTS

The Workouts consist of multi-step problems that often require students to use several pieces of their mathematical knowledge. These problems can be used in the classroom to challenge students and to extend their thinking. The Workouts can be used to prepare students for the Target and Team Rounds of competition.

Answers to the Workouts include one-letter codes, in parentheses, indicating appropriate problem-solving strategies. However, students should be encouraged to find alternative methods of solving the problems; their methods may be better than the one provided! The following strategies are used: **C** (Compute), **F** (Formula), **M** (Model/Diagram), **T** (Table/Chart/List), **G** (Guess & Check), **S** (Simpler Case), **E** (Eliminate) and **P** (Patterns).

MATHCOUNTS Symbols and Notation

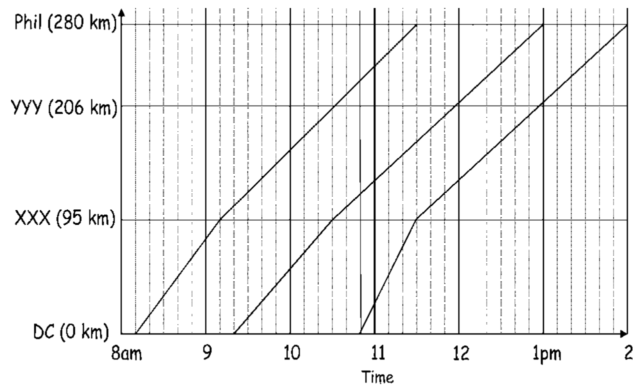
Standard abbreviations have been used for units of measure. Complete words or symbols are also acceptable. Square units or cube units may be expressed as units^2 or units^3.

WORKOUT 1

1. ________ If x is 150% of y, what percent of 3x is 4y? Express your answer to the nearest whole number.

2. ________ The ratio of the number of cans of cola soda to lemon-lime soda to cherry soda consumed at a graduation party was 12:3:10. If a total of 150 cans of these three flavors of soda were consumed, how many cans were lemon-lime soda?

(For #3 - #5) The graph depicts the schedule of three trains from Washington, DC to Philadelphia. The y-axis indicates the distance in kilometers of each city from Washington, DC.



3. ________ Tomas left DC at 8:10 am, stopped to shop in XXX and then caught the next train to Philadelphia (Phil). How many total minutes did Tomas spend riding on the trains?

4. ________ What is the average speed, in kilometers per hour, of the train from DC to Philadelphia that leaves DC at 9:20 am? Express your answer as a decimal to the nearest tenth.

5. ________ What is the average speed, in kilometers per hour, of the fastest train between XXX and Philadelphia? Express your answer as a decimal to the nearest tenth.

6. ________ What is $\frac{1}{5}$% of 140? Express your answer as a decimal to the nearest hundredth.

7. ________ A goat is attached to an L-shaped rod with a leash that allows the goat to move a ground distance of 8 meters from the rod on all sides. AB = 10 m, BC = 20 m and AB is perpendicular to BC. The attached end of the leash may move along the entire rod and the goat may roam on all sides of the rod. What is the number of square meters in the area of the region of grass that the goat can reach? Express your answer in terms of π.

8. ________ What is the number of square units in the area of a triangle whose sides are 5, 6 and $\sqrt{13}$ units. Express your answer in simplest form.

9. ________ The product of a set of distinct whole numbers is 120. What is the least possible sum of the members of the set?

10. ________ If a = 4.9 and b = $1\frac{5}{9}$, determine the value of the reciprocal of b/a . Express your answer as a common fraction.

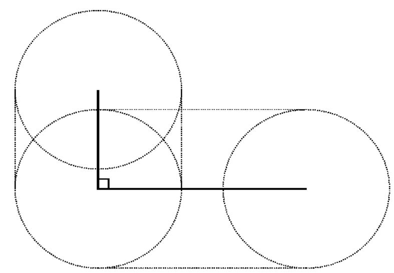
WORKOUT 1

Answers

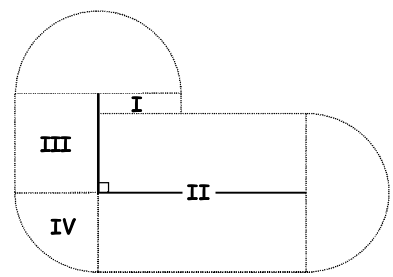
1.	89	(C)	5.	79.3	(F, T)	8.	9	(F)
2.	18	(F, S)	6.	.28	(C)	9.	14	(G, T)
3.	210	(F, T)	7.	$416+80\pi$	(M, C)	10.	$\frac{63}{20}$	(C)
4.	76.4	(F, T)						

Solution — Problem #7

This problem is similar to the triangular sandbox/fence problem seen in an earlier Warm-up. We need to keep in mind that the goat's leash will allow him to move a ground distance of up to 8 meters from any given point on the L-shaped rod. The first picture shows all of the points that are 8 meters from the segment portions of the rods (these are the segments running parallel to the rods on either side of them) and all of the points that are 8 meters from the tips of the rods (these are the circles at each vertex/endpoint).



To show the outline of the actual region the goat can move in, we can look at the second figure. Some of the extra segments within the boundaries have been taken out, while some other segments have been added. Notice the goat's roaming space is now divided into three rectangles (I, II and III) and three portions of circles that we can now find the area of.



Region I: 2x8 = 16 square meters
Region II: 20x16 = 320 square meters
Region III: 10x8 = 80 square meters
Region IV: $(1/4)\pi 8^2 = 16\pi$ square meters
Both semicircles = One circle = $\pi 8^2 = 64\pi$ square meters
TOTAL = $416 + 80\pi$ square meters

Connection to... Uses of Graphs (Problem #5)

This graph is similar to a simplified version of train schedules used in France in the late 1800's. All of the daily trains between Paris and Dijon, including all stops, were represented on one graph. The trains from Paris to Dijon were represented by line segments with positive slopes while the return trains from Dijon to Paris were represented by line segments with negative slopes. The steepness of the slopes indicated the average speeds of the trains. This format made it easy for travelers to plan their day's itinerary. (Reference: "Elementary and Intermediate Algebra", Second Edition, Bittinger, Ellenbergen, and Johnson, Page 151)

Investigation & Exploration (Problem #8)

Let's call a lattice triangle any triangle whose vertices have integer coordinates. Notice that the side lengths of such a triangle need not be integers. Can you find a lattice triangle for which (a) all three side lengths are integers, (b) exactly two side lengths are integers, (c) exactly one side length is an integer and (d) none of the side lengths are integers?

WORKOUT 2

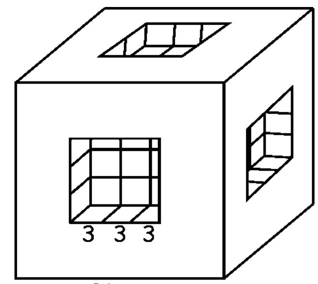
1. ________ The cost of a new car, including *n*% sales tax, was $20,276.50. The cost before sales tax was $18,950. What was the value of *n*?

2. ________ Between 10:50 AM and 1:30 PM, Bill rides 42 miles on his bicycle. What is his average speed in miles per hour? Express your answer as a decimal to the nearest hundredth.

3. ________ What is the total number of digits used when the first 2002 positive even integers are written?

4. ________ Kelly's average score on four Spanish tests is 85.5. The average of her three highest scores is 87, and her two lowest scores are the same as each other. What is the average of her two highest test scores?

5. ________ A solid cube measures 21 cm on an edge. Nine cubes of edge 3 cm are removed from the center of each face of the original cube. What is the number of square centimeters in the surface area of the new object?



6. ________ Benson has Golden Delicious apples, each of which weighs .6 pounds, and Jonathan apples, each of which weighs .8 pounds. He wants to make applesauce such that 1/3 of the weight is from Golden Delicious apples and 2/3 of the weight is from Jonathan apples. He wants to use all 12 of his Golden Delicious apples. How many Jonathan apples should he use?

7. ________ Tyrone announces, "I just found $5.00. I now have five times more money than if I had lost $5.00." How many dollars did Tyrone have before finding the $5.00? Express your answer to the nearest hundredth.

8. ________ A 5 x 8 rectangle can be rolled to form two different cylinders with different maximum volumes. What is the ratio of the larger volume to the smaller volume? Express your answer as a common fraction.

9. ________ A "palindrome" is a positive integer that reads the same backwards and forwards. For example, 727 and 888 are palindromes. What is the largest 4-digit palindrome which is the sum of 2 different 3-digit palindromes?

10. ________ N is a natural number such that $2^x > x^8$ for all $x > N$. What is the minimum value for N?

WORKOUT 2

Answers

1.	7	(F, C)	5.	3294	(P, M)	8.	8/5	(F)
2.	15.75	(F, C)	6.	18	(F, C)	9.	1221	(E, P)
3.	7456	(P)	7.	7.50	(F)	10.	44	(E, G)
4.	90	(T, E, F)						

Solution — Problem #9

Let's take what we know about addition problems and see if we can figure this puzzle out! We've been given the following addition problem:

$$\begin{array}{cccc} & A & B & A \\ + & C & U & C \\ \hline E & F & F & E \end{array}$$

E is the "Carry Digit" from A + C, therefore, E = 1. Since we are only adding together two digits, their sum can't be more than 19, even assuming a 1 was carried over from adding the tens column. Now we know that the units column, A + C, must equal 11.

Since A + C equals 11 in the units column, they must also equal 11 for the hundred's column. Therefore, F = 1 or 2 (if a 1 has been carried over from the tens column). So the largest that EFFE can be is 1221.

Many combinations yield N = 1221, for example: 787 + 434 = 1221. Can you find others?

Connection to... World Population (Problem #10)

2^x is an example of "exponential growth" and x^8 is an example of "polynomial growth." For all sufficiently large values of x, exponential growth (b^x , b>1) always exceeds polynomial growth (x^k , k>1).

In the late 1700's, the English political economist, the Reverend Thomas Malthus, averred that world population growth is exponential while the increase in food production is polynomial (linear, in fact). He therefore concluded that the English economy would face increasing problems in trying to provide relief for the poor. The Reverend Malthus' predictions have not come true because over the last 200 years, food production has increased exponentially due to fertilizer, technology and improved methods of farming. However, Malthus did provide one of the earliest examples of using mathematical concepts to forecast the future and to address social issues. (References: http://www.cs.hmc.edu/~belgin/Population/malthus.html ; http://www.stolaf.edu/people/mckelvey/envision.dir/malthus.html)

Investigation & Exploration (Problem #8)

Explore rectangles that have dimensions a x b. For example, for what values of a and b will the ratio of the volumes be 2:1? What generalization can be made?

WORKOUT 3

1. ________ A pendulum oscillates with period, P, such that P = $2\pi\sqrt{\frac{L}{g}}$ and $g = 9.8$ m/s^2 and L is the number of meters in the length of the pendulum. What is the number of meters in the length of a pendulum with a period of one second? Express your answer as a decimal to the nearest hundredth.

2. ________ Sixty-one percent of the world's population live in Asia. Of the remainder, 14% live in South America and 49% of all South Americans live in Brazil. What percent of the world's population lives in Brazil? Express your answer as a decimal to the nearest tenth.

3. ________ Calculate $\frac{1}{4} \bullet \frac{2}{5} \bullet \frac{3}{6} \bullet \frac{4}{7} \bullet \ldots \bullet \frac{49}{52} \bullet \frac{50}{53}$. Express your answer as a common fraction.

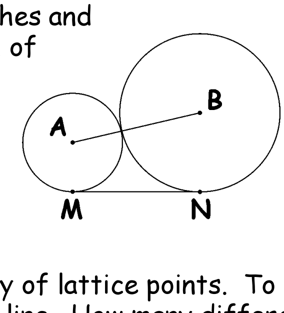
4. ________ A car traveled at an average rate of 66 feet per second for 160 minutes. How many miles did the car travel?

5. ________ Heron's formula (sometimes called the semi-perimeter formula) says that if a triangle has side lengths a, b and c, then the area of the triangle is given by A= $\sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{(a+b+c)}{2}$. As a decimal to the nearest tenth, how many square inches are in the area of the triangle with sides of length 4, 5 and 6 inches?

6. ________ What is the number of square centimeters in the area of a semicircular region with a perimeter of 20 cm? Express your answer as a decimal to the nearest tenth.

7. ________ The speed of light is 670,000,000 miles per hour. How many seconds does it take light to travel 121,000,000 miles? Express your answer to the nearest second.

8. ________ Circles A and B are externally tangent and have radii 9 inches and 16 inches, respectively. How many inches are in the length of the common tangent MN?



9. ________ Three dimensional tic-tac-toe is played on a 3 x 3 x 3 array of lattice points. To win, you must choose three points which lie along the same line. How many different ways can such a line be formed?

10. _______ The proper divisors of 12 are 1, 2, 3, 4 and 6. A proper divisor of an integer N is a positive divisor of N that is less than N. What is the sum of the proper divisors of the sum of the proper divisors of 284?

WORKOUT 3

Answers

1. .25 (F)	4. 120 (F)	8. 24 (F, M)
2. 2.7 (C, F)	5. 9.9 (C, F)	9. 49 (P)
3. $\frac{1}{23,426}$ (C)	6. 23.8 (F, M)	10. 284 (P, E)
	7. 650 (C, F)	

Solution — Problem #9

In order to count the lines, we can classify them by the location of their midpoint.

• Suppose the midpoint is on an edge of the cube. There are 12 edges of the cube, each of which gives one such line.

• Suppose the midpoint is the center of a face of the cube. There are 6 such points, each of which is the middle of 4 different lines, yielding 24 total lines.

• Suppose the midpoint of the line is the centermost point of the grid. There are 26 other points and each of the 13 pairs of opposite points defines a line, which passes through the centermost point.

Thus, there are 12 + 24 + 13 = 49 ways such lines can be formed.

Connection to... Astronomy (Problem #7)

The speed of light is so fast that in the 17^{th} Century astronomers were unable to measure its speed and some hypothesized light was infinitely fast. Based on an extensive study of astronomical data on sightings of Jupiter and its moon Io, the 21-year old Danish astronomer Ole Roemer had formed the hypothesis that the speed of light was approximately 670 million miles per hour and that, based on the location of Earth and Jupiter in their orbits around the sun, the light from Io would have to travel 121 million miles farther than the last time the same measurements were made. Based on these hypotheses, in 1671 Ole Roemer predicted that the next sighting of Jupiter's moon, Io, would be 10 minutes and 50 seconds (that is, 650 seconds) later than predicted by the prominent astronomer Cassini. Even though Roemer's prediction was proven correct, eminent astronomers refused to accept his hypotheses until another 50 years had passed. (Reference: "A Biography of the World's Most Famous Equation – $E = mc^2$ " by David Bolanis.)

Investigation & Exploration (Problem #10)

The numbers 220 and 284 are called an amicable pair of numbers. Each is the sum of the proper divisors of the other. Show that 1184 and 1210 is also an amicable pair. Next try this method for generating amicable pairs.

Let $a = 3 \cdot 2^n - 1$, $b = 3 \cdot 2^{n-1} - 1$ and $c = 9 \cdot 2^{2n-1} - 1$. Suppose all three of a, b and c are prime numbers. Then the pair $2^n \cdot a \cdot b$ and $2^n \cdot c$ is an amicable pair. For n=4 we get that a=47, b=23 and c=1151, which are all primes, so $2^4 \cdot 23 \cdot 47$ and $2^4 \cdot 1151$ is an amicable pair. Check this out by finding the sum of their proper divisors.

WORKOUT 4

1. ________ Miss Johnson has m guests over for a cookout. She makes 24 hamburgers, and all guests receive the same number of burgers. How many possible values are there for m if Miss Johnson has more than one guest over?

2. ________ The sum of 40 consecutive integers is 100. What is the largest of these 40 integers?

3. ________ What is the largest integer k such that $k^4 < 10^6$?

4. ________ The UPC code, made up of numbers and dashes, on a video tape is 9-78094-11006-x. The digit x is in the 12th position. Let n be the value obtained by adding the digits in the odd positions, tripling that sum, and then adding the digits in the even positions. A valid UPC code is such that n is divisible by 10. For what value of x will this UPC code be valid?

5. ________ The product of three consecutive positive integers is 10,626. What is the sum of the three integers?

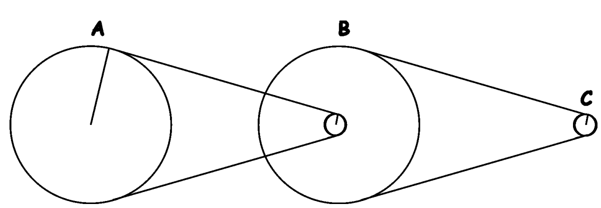
6. ________ City Cab Company charges \$1.60 plus \$0.25 per 1/8 mile. The distance from the airport to the Ritz hotel is 13.25 miles. Two passengers will share the fare equally. How many dollars will each passenger owe? Express your answer to the nearest hundredth.

7. ________ Points A(12,0), B(0,16) and C(10,10) are connected to form a triangle. From the six points determined by A, B, C and the midpoints of each of the sides of the triangle, what is the number of units in the shortest distance from any of these six points to the origin?

8. ________ Dot has a bag of apples and each apple weighs exactly 0.7 pounds. When the apples are placed on the scale, the scale shows .3 pounds, because, while the tenths digit lights up correctly, the lights recording the number of pounds are malfunctioning. What is the fewest number of apples that could be in the bag?

9. ________ Two horses on a merry-go-round are placed 8 and 17 feet from the center of the circular path they follow. The horses make one complete rotation in nine seconds. What is the positive difference, in feet per second, between the average speeds of the horses? Express your answer as a decimal to the nearest tenth.

10. ________ Wheels A, B and C are attached by belts as shown, and the two parts of wheel B are connected and turn together as one wheel. The radii of the two larger wheels are 6 inches and the radii of the two smaller wheels are 1 inch. How many revolutions will wheel C make while wheel A makes one revolution?



WORKOUT 4

Answers

1.	7	(P, T, C)	5.	66	(C, E, G)	8.	9	(P, T)
2.	22	(P, C, G)	6.	14.05	(C)	9.	6.3	(F, M)
3.	31	(E, C, G)	7.	10	(F, M)	10.	36	(P, F, C)
4.	9	(C)						

Solution — Problem #1

The question really asks for the number of positive integer divisors of 24. We could list the possibilities: 1, 2, 3, 4, 6, 8, 12 and 24. Remember, we know she does not have just one guest, so there are **7 possible values** for m. For numbers larger than 24, though, listing the factors could be cumbersome. It would be nice to have a method for counting the divisors of an integer, without having to list all of them. Each divisor is the product of some of the prime factors of 24, so begin by prime factoring 24 as $2^3 \times 3$. Then, each divisor is of the form $2^a \times 3^b$, where a is either 0, 1, 2, or 3 (4 possibilities) and b is either 0 or 1 (2 possibilities). This implies that there are $4 \cdot 2 = 8$ factors of 24. The factors are shown in the table below:

	2^0	2^1	2^2	2^3
3^0	1	2	4	8
3^1	3	6	12	24

Connection to... UPC codes (Problem #4)

Universal Product Codes (UPCs) are constructed so that the first digit represents the type of product, the next five digits identify the manufacturer, the five after that label the specific product and the final digit is a "check digit." A computer scanner can then make the computation described in this problem. If the result isn't divisible by 10, the computer knows that it has scanned the numbers incorrectly. Test the UPC code on some products around your home or school.

This type of check digit is also designed to detect transpositions, or the switching of two adjacent digits. Try swapping two digits to see if the number still satisfies the checking criteria.

Investigation & Exploration (Problem #8)

This problem does not focus on whole numbers of pounds. Instead it focuses on the remainders. This type of arithmetic is called "modular" arithmetic; in this problem, we are using mod 10. The equation we're trying to solve, then, is $7n = 3 \pmod{10}$. In regular algebra, we'd solve an equation like this by multiplying by (1/7). In mod 10 arithmetic, though, that would give us an answer of 3/7, which isn't really what we mean. Instead, notice that $3 \cdot 7$ is 21, which has a units digit of 1, or a remainder of 1 when divided by 10. This means that 21 = 1 in mod 10. Thus, multiplying both sides of the equation by 3 yields $21n = n = 3 \cdot 3 = 9$, so n is 9.

WORKOUT 5

1. ________ Four thousand young people are attending a rock concert. Five percent are wearing exactly one earring. Fifty percent of the other ninety-five percent are wearing exactly two earrings. Ten percent of the remaining people are wearing exactly three earrings. The rest of the people are not wearing any earrings. What is the total number of earrings being worn at the concert?

2. ________ How many three-digit numbers contain the digit 3 at least once?

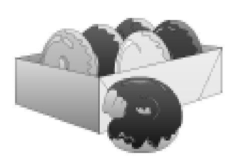
3. ________ The hypotenuse of a right triangle is twice the length of one leg of the triangle. The length of the other leg is 12 cm. How many square centimeters are in the area of the right triangle? Express your answer in simplest radical form.

4. ________ Using each of the digits 2 through 9, one per square, what is the maximum value of the following expression?

$$\square\,\square\,\square\,\square\ +$$
$$\left(\square\,\square \times \square\,\square\right)$$

5. ________ A square and an equilateral triangle are inscribed in a circle. What is the ratio of the area of the triangle to the area of the square? Express your answer as a fraction in simplest radical form.

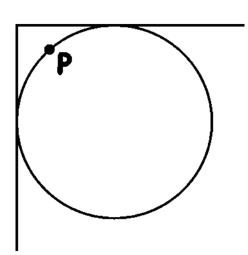
6. ________ Dora's Delicious Doughnuts made its first batch of doughnuts one Monday morning at 8am, and has continued to make fresh doughnuts every five hours ever since. How many weeks will it be before Dora's Delicious Doughnuts makes fresh doughnuts on a Monday morning at 8am again?



7. ________ What integer is closest to the value of $\sqrt[3]{6^3 + 8^3}$?

8. ________ What is x, if $x^{12} = 2$? Express your answer as a decimal to the nearest hundredth.

9. ________ A circular table is tangent to two adjacent walls of a rectangular room. Point P, on the edge of the table, is 12 inches from one wall and 16 inches from the other wall as shown to the right. What is the number of inches in the diameter of the table? Express your answer to the nearest whole number.



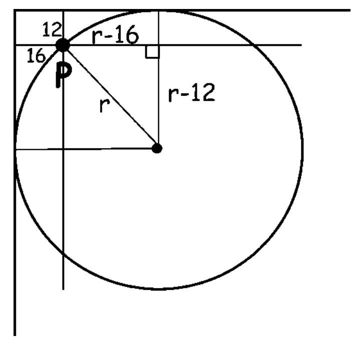
10. ________ Four standard 6-sided dice are rolled. The product of the four numbers rolled is 144. How many different sums of four such numbers are possible?

WORKOUT 5

Answers

1.	4570	(C, F)	5.	$\frac{3\sqrt{3}}{8}$	(F, M)	8.	1.06	(C, E)
2.	252	(T, P, E)	6.	5	(P, F)	9.	95	(M, F)
3.	$24\sqrt{3}$	(F)	7.	9	(C)	10.	4	(T)
4.	15,932	(P, E, G)						

Solution — Problem #9



In the diagram we can see where P is 12″ and 16″ from the walls, and three radii have been drawn in, one to P and one to each of the walls perpendicularly. There are also two secants drawn in the picture which each run parallel to a drawn radius. Notice the right triangle that is formed. Using the Pythagorean theorem, we see that $r^2 = (r-16)^2 + (r-12)^2$. After multiplying out each squared binomial, combining like terms and moving each term to the right, we have $0 = r^2 - 56r + 400$.

One way to solve this quadratic is with the quadratic formula which says:

$$x= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{56 \pm \sqrt{(-56)^2 - 4(1)(400)}}{2(1)} \approx 47.595918 \text{ or } 8.4040821$$

Notice that our second value of x is not possible for this drawing. Also remember that we are looking for the diameter, so we must multiply our answer for x by 2, giving us approximately 95 inches.

Solution — Problem #10

The prime factorization of 144 is $2^4\,3^2 = 2{\cdot}2{\cdot}2{\cdot}2{\cdot}3{\cdot}3$, so dice rolls of 1, 2, 3, 4 and 6 must be considered.

It is often easiest to solve counting problems by breaking them into cases:

Two sixes: If two sixes are rolled, their product will be 36. So 144/36 = 4. The only ways to get a product of 4 from two remaining dice are 2,2 or 1,4. Therefore, two sums come from 6,6,2,2 and 6,6,1,4.

One six: If one six is rolled, we know 144/6 = 24. The only way to get a product of 24 from three dice (without using any more 6's) is 3,4,2. The third sum is from 6,3,4,2.

No sixes: The only way to get a product of 144 without rolling any sixes is 3,3,4,4, giving us the fourth and last possible sum.

Connection to... Music (Problem #8)

Each note of the musical scale is characterized by a particular frequency. The A above middle C on a piano, for example, is 440 Hz, or cps (cycles per second). The piano keyboard has a repeating pattern of 12 keys - 7 white and 5 black - and the ratio between every two consecutive keys/notes was chosen so that the ratio between octaves would be exactly two. For example, the next higher A has frequency 880 Hz. The value of x in this problem is the ratio between two consecutive notes. Using your value of x, find the integer value of m so that x^m is closest to 4/3. If you play two notes that are m apart, their tones sound good together. Find the integer value of n so that x^n is closest to 3/2. Notes that are n apart are also pleasing to the ear.

There are many ways that math ties into music. Even Pythagoras, whom we generally associate with right triangles, did a lot of work examining the relationships between notes. Check out the "Online Math Applications" at http://tqjunior.thinkquest.org/4116/Music/music.htm for more information on the many ties between math and music!

WORKOUT 6

1. ________ The number $-\frac{63}{5}$ is between two consecutive odd integers. What is the product of these two integers?

2. ________ For what value of n, where n is the units digit, is 234,56n divisible by 7?

3. ________ Given: $\frac{5}{13}=\frac{n}{39}=\frac{m+n}{156}=\frac{p-m}{104}$. What is the value of p?

4. ________ A cylindrical quarter has a $\frac{15}{16}$ inch diameter and a $\frac{1}{16}$ inch height. What would be the number of inches in the height of a coin whose volume is exactly four times that of the given quarter and whose diameter equals $1\frac{1}{8}$ inches? Express your answer as a common fraction.

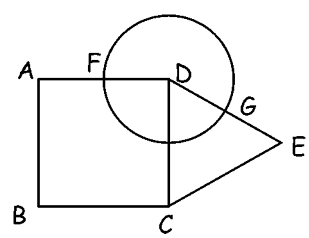
5. ________ What is the largest four-digit number, the product of whose digits is 6! ?

6. ________ What is the x-intercept of the line perpendicular to the line defined by 3x-2y=6 and whose y-intercept is 2?

7. ________ What is the product of all integer perfect squares less than 50?

8. ________ How many four-digit numbers have the property that each of the three two-digit numbers formed by consecutive digits is divisible either by 19 or 31?

9. ________ A circle with diameter 2 cm is centered at a vertex D of the square and intersects square ABCD and equilateral triangle DCE at midpoints F and G, respectively. What is the number of square centimeters in the area of the region obtained by taking the union of the interiors of the three figures? Express your answer as a decimal rounded to the nearest hundredth.



10. ________ The different toppings available at Conway's Ice Cream Parlor are given below. A customer walks up and says, "I'd like a scoop of chocolate ice cream with any 2 different wet toppings and any 3 different dry toppings. Surprise me!" How many different combinations of toppings are possible for the customer's order?

Wet Toppings
Caramel
Fudge
Chocolate Syrup
Butterscotch

Dry Toppings
M&M's
Heath Bar
Butterfinger
Peanuts
Sprinkles
Gummi Bears
Oreos
Nestle Crunch
Pecans

WORKOUT 6

Answers

1.	143	(C, E, M)	5.	9852	(P)	8.	8	(P, E)
2.	3	(E, P)	6.	3	(F)	9.	7.56	(F)
3.	85	(C)	7.	25,401,600	(C)	10.	504	(P, S)
4.	$\frac{25}{144}$	(F)						

Solution — Problem #8

First we need to know which 2-digit numbers are divisible by either 19 or 31. For 19 we have the following options: 19, 38, 57, 76 and 95. The multiples of 31 are 31, 62 and 93. So we are looking for as many 4-digit numbers as we can find whose three 2-digit numbers formed by consecutive digits are from the list **19, 31, 38, 57, 62, 76, 93** and **95**.

Let's see if we can find a number that starts with the 19. It will be in the form 19_ _. There are two options for the third digit since 93 and 95 are in our list, so we have 193_ and 195_. The first number can be finished with either a 1 or an 8 since 31 and 38 are in our list and the second number can be finished with a 7 since 57 is in the list. So we have three numbers that start with 19: 1931, 1938 and 1957.

Start with 31_ _ --> 319_ --> 3193 or 3195.

Now start with 38_ _. There are no numbers in our list that start with an 8, so we're finished.

Let's continue the pattern. 57_ _ --> 576_ --> 5762

62_ _ --> Finished...no numbers starting with a 2.

76_ _ --> 762_ --> Finished...no numbers starting with a 2.

93_ _ --> 931_ and 938_ --> 9319 and the second one is finished...no numbers starting with an 8.

95_ _ --> 957_ --> 9576

We have found a total of 8 numbers that fit the requirements.

Investigation & Exploration (Problem #7)

For Problem #7, we can see that the set of all the natural number perfect squares less than 50 is {1, 4, 9, 16, 25, 36, 49} = {1•1, 2•2, 3•3, 4•4, 5•5, 6•6, 7•7}. The product of these numbers can be shown in the following form: 1•1•2•2•3•3•4•4•5•5•6•6•7•7 which is $(7!)^2$ or $(5040)^2$ = 25,401,600.

From this problem, can you figure out a quick way to show the product of all the natural number perfect cubes less than 217? What would be the product of all the natural number perfect cubes less than $n^3 + 1$?

WORKOUT 7

1. ________ One-fourth of Holttown High School's students are seniors, one-third are juniors, and the other 300 students are sophomores. Of the seniors, two-fifths are boys. How many senior girls are students at Holttown High School?

2. ________ In the game of golf, par is the term used to describe the number of shots it should take for a professional golfer to get the ball in the hole. The par scores and Tiger Woods' scores for the last 9 holes of the 2000 Professional Golfers Association Championship are shown. How many shots below par was Tiger Woods for these 9 holes?

Hole Number	Par Score	Tiger Woods
10	5	4
11	3	3
12	4	3
13	4	4
14	3	2
15	4	4
16	4	4
17	4	3
18	5	4

3. ________ What is the maximum integer value of n such that 2^n is a factor of 120! ?

4. ________ The five interior angle measures of a pentagon are 2x, 3x, 4x, 5x and 6x degrees. The measures of their corresponding exterior angles are a, b, c, d and e degrees, respectively. What is the value of the largest ratio: a/b, b/c, c/d, d/e or e/a ? Express your answer as a common fraction.

5. ________ Justin is reducing the number of cans of soda he consumes each day. After today, he will wait a full day before having another. Then he'll wait two more days, then three, and so on, extending his waiting period by one day each time. In how many years (to the nearest year) will he be drinking a can of soda only once every 60 days?

6. ________ Call a set of positive integers a "phancy set" if the product of any two integers in the set is 1 less than a perfect square. What is the least possible value for n such that {4,6,n} is a phancy set?

7. ________ Three 3-digit numbers are formed using the digits 1 through 9 exactly once each. The hundreds digit of the first number is 1. The tens digit of the second number is 8. The units digit of the third number is 5. The ratio of the first number to the second number to the third number is 1:3:5, respectively. What is the sum of the three numbers?

8. ________ What is the maximum number of 3" x 4" rectangles that will fit, without overlap, within a 20" x 20" square?

9. ________ The product of the digits of a four-digit number is 6!. How many such 4-digit numbers are there?

10. ________ The following table indicates the fuel consumption, in gallons/hour, for a car traveling at various speeds. At which of these speeds, in miles per hour, does the car consume the least gallons of gasoline per mile?

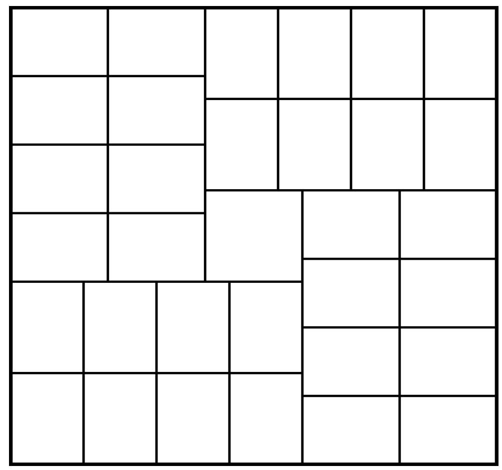
speed (miles/hour)	10	20	30	40	50	60
gasoline used (gallons/hour)	.90	1.20	1.40	1.70	2.00	2.50

WORKOUT 7

Answers

1.	108	(C)	5.	5	(P)	8.	33	(P, M)
2.	5	(C)	6.	20	(E)	9.	72	(P, E)
3.	116	(P)	7.	1161	(P, E)	10.	50	(F)
4.	5/2	(F)						

Solution — Problem #8



One might start by recognizing that 20 is 5x4 so 5 rows of rectangles could be arranged in 6 three-inch columns using 30 rectangles. This would leave a 2x20 inch strip left over. Recognizing that 3+3+2 = 8 suggests that it would be possible to rotate the last two rows of rectangles and fit in more. This arrangement uses 32 rectangles with a 2x8 rectangular strip left over. This strip has an area of 16 squares which is 4 more than the area of the 3x4 rectangle. Is it possible to rearrange the rectangles to use the remaining area? Consider a 12x8 rectangle made from eight 3x4 rectangles and arrange them as we did in the figure to the right.

This leaves a 4x4 square in the middle where one more 3x4 rectangle may be placed, using a total of 33 rectangles!

Solution and Investigation & Exploration (Problem #3)

The number 120! can be thought of as being the string of factors 1•2•3•4•...•119•120.

Sixty of these factors are divisible by 2. (Giving us 60 factors of 2.)

Thirty of these factors are divisible by 2^2. (Giving us another 30 factors of 2.)

Fifteen of these factors are divisible by 2^3. (Giving us another 15 factors of 2.)

Seven of these factors are divisible by 2^4. (Giving us another 7 factors of 2.)

Three of these factors are divisible by 2^5. (Giving us another 3 factors of 2.)

One of these factors is divisible by 2^6. (Giving us another factor of 2.)

None of these factors are divisible by 2^7.

Thus 60 + 30 + 15 + 7 + 3 + 1 = 116 is the total number of two factors of 120!

What if the question was changed to: What is the maximum integer value of n such that 5^n is a factor of 120! ? Or 10^n? How many trailing zeros are there if 120! is multiplied out?

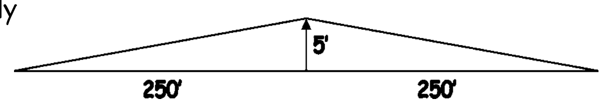
Connection to... Golf (Problem #2)

Golf is a sport in which positive and negative numbers play a role. The term, *par*, is the number of strokes a skillful player is expected to take to get the ball into the hole. If the par for a particular hole is 5, and a player gets the ball in the hole in just 3 shots, the score for the hole would be –2 because it was made in two fewer shots than expected. If the player makes it in 6 shots, the score for the hole would be +1 because it was made in one more shot than expected. There are special names for some scores such as bogey, birdie and eagle. You might like to look up their meanings.

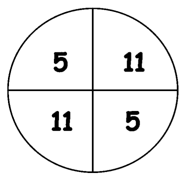
WORKOUT 8

1. ________ The product ab = 1200 and b is an odd number. What is the largest possible value of b?

2. ________ A flat steel bridge is built from two rigid 250 foot long beams joined at the middle. On a hot day, the beams expand equally causing the joint to rise 5 feet. By how many inches did one of the beams expand? Express your answer as a decimal to the nearest tenth.

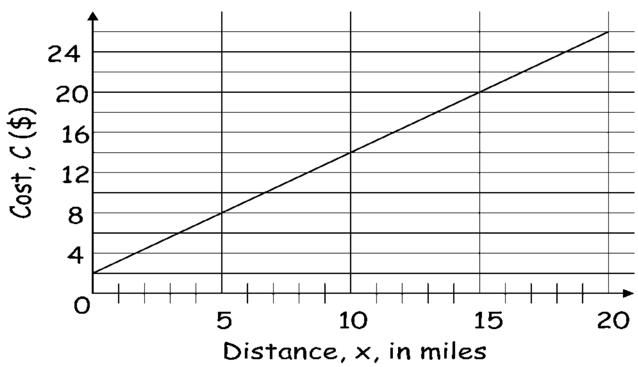


3. ________



An unlimited number of darts are to be thrown at a dart board with possible scores as shown to the left. What is the greatest whole number score that is not possible to achieve?

(For #4 - #6) The graph represents the cost C, in dollars, of a taxi ride of distance x, in miles.



4. ________ A sign advertises the cost of a ride as having an initial fee of \$a, plus \$b per mile. Calculate b. Express your answer as a decimal to the nearest hundredth.

5. ________ If the graph continues as a straight line, what is the number of dollars in the cost of a 35-mile ride?

6. ________ If the graph continues as a straight line, what is the number of miles in the length of a ride that costs \$53.60?

7. ________ Define the function a @ b = a(b) + b. What is the value of 1@(1@(1@(1@(1@1))))?

8. ________ The points A, B and C lie in a plane and have coordinates (6,5), (2,1) and (0,k), respectively. What value of k makes the sum of the lengths of segments AC and BC the least possible value?

9. ________ What is the least whole number with exactly eleven factors?

10. _______ Find all 6-digit multiples of 22 of the form 5d5,22e where d and e are digits. What is the maximum value of d?

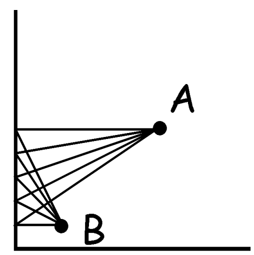
WORKOUT 8

Answers

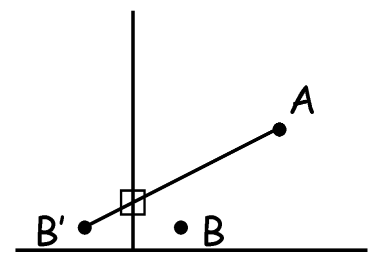
1.	75	(E, P)	5.	44	(*C*)	8.	2	(E, F, M)
2.	0.6	(F)	6.	43	(*C*)	9.	1024	(P)
3.	39	(P)	7.	32	(*C*)	10.	8	(E, *C*, P)
4.	1.20	(F)						

Solution — Problem #8

Because the ordered pair for point *C* is in the form (0,k), we know that the point must be somewhere along the y-axis. The first diagram shows the segments for five different placements of *C*. We need to find the pair of segments with the shortest combined distance.



The problem is equivalent to one obtained by reflecting (2,1) in the y-axis (as shown in the second diagram). We can see that the shortest path from (-2,1) to (6,5) is a straight line.



We can find the slope of this line since we know two points on the line determine the slope, which is given by the following formula:

$$\frac{\textit{change in y}}{\textit{change in x}} = \frac{5-1}{6--2} = \frac{4}{8} = \frac{1}{2}$$

Using the slope intercept form of a linear equation, we have $y=(^1/_2)\,x + k$. Substituting (-2,1) for *x* and *y*, we get $1= (^1/_2)(-2) + k$, and k = 2. So point *C*, found in the box in the second diagram, is (0,2).

Connection to... Engineering (Problem #2)

Engineers need to know how different materials will expand and contract as a function of temperature when they design buildings and highways. Investigate how different materials change as a function of temperature. Investigate how long bridge spans can be without creating gaps that are too large for a vehicle to pass over.

Investigation & Exploration (Problem #7)

In general, what does 1@n do to n? What's another way to write 1@1@…@1 n times? What about 2@n? Does the position of the parentheses affect the outcome? For instance, would we get the same answer for (((((1@1)@1)@1)@1)@1)? If the parentheses don't affect the answer, then we say that the operation @ satisfies the Associative Property. Does @ satisfy the Associative Property?

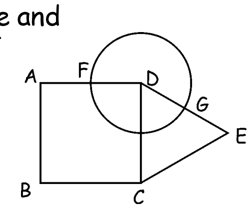
WORKOUT 9

1. ________ By how many degrees does the measure of an interior angle of a regular decagon exceed the measure of an interior angle of a regular pentagon?

2. ________ A small hose fills a swimming pool in 16 hours. A large hose connected to a different water supply fills the same pool in 12 hours. With the pool empty, the owner turns on the smaller hose at 8:00am. He turns on the larger hose at 10:00am. Both hoses are used from 10:00am to 3:00pm. What percent of the pool is full at 3:00pm? Express your answer to the nearest tenth.

3. ________ To test whether an integer, n, is prime, it is enough to be sure that none of the primes less than the square root of n divide n. If you want to check that a number between 900 and 950 is prime with this rule, what is the largest prime divisor you need to test?

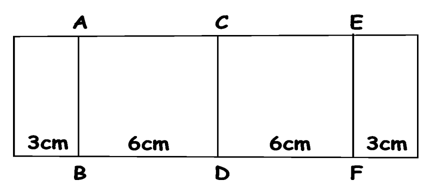
4. ________ A circle with diameter 2 cm is centered at a vertex D of the square and intersects square ABCD and equilateral triangle DCE at midpoints F and G, respectively. What is the number of centimeters in the perimeter of the region obtained by taking the union of the interiors of the three figures? Express your answer as a decimal to the nearest hundredth.



5. ________ What is the sum of all of the multiples of 3 between 100 and 200?

6. ________ For how many positive integers p does there exist a triangle with sides of length $3p - 1$, $3p$, and $p^2 + 1$?

7. ________ A quarter, 2.5 centimeters in diameter, is dropped randomly on the tabletop shown so that at least half of the coin lies on the tabletop. What is the probability that the quarter lies on one of the segments: $\overline{AB}$, $\overline{CD}$ or $\overline{EF}$? Express your answer as a common fraction.



8. ________ What is the sum of all the elements of the two-element subsets of {1, 2, 3, 4, 5, 6} ?

9. ________ Allison has sneezed exactly one million times in her life. Because of her age there has to have been at least one day when she sneezed at least 101 times. What is the oldest, in days, Allison could be?

10. _______ If n is an integer and $20 < 2^n < 200$, what is the sum of all of the possible values of n?

WORKOUT 9

Answers

1.	36	(F)	5.	4950	(P, C, F)	8.	105	(P, C, T)
2.	85.4	(F, M)	6.	5	(G, P)	9.	9999	(S, F)
3.	29	(E)	7.	$\frac{5}{12}$	(M, S)	10.	18	(E)
4.	11.67	(F)						

Solution — Problem #9

The solution to this problem involves a useful mathematical principle called the "Pigeon Hole Principle" (PHP). In its simplest form, the PHP says that if you have more pigeons than pigeon holes, then there must be some pigeon hole which has more than one pigeon! In its more general formulation, the PHP says that if you have nm + 1 pigeons in n pigeonholes, then there is some pigeon hole with more than m pigeons. To apply the pigeon hole principle to this problem, suppose that Allison has never sneezed as much as 101 times in a day. Then she must have lived at least 1,000,000 (sneezes)/100 (sneezes per day) = 10,000 days. If she sneezed fewer than 100 times in a given day, then she would have had to live even longer! But, if she's lived fewer than 10,000 days, then there must have been a day when she sneezed at least 101 times. To make this conclusion, therefore, she must have lived no more than 9,999 days.

Connection to... Computer Security (Problem #3)

Eratosthenes was a Greek mathematician (ca 284-192 B.C) well known for his "Sieve." A sieve is the kitchen item you use to sift flour. Eratosthenes' Sieve was a filter for prime numbers. He began with all the positive integers, first eliminating all the multiples of 2, then the multiples of 3 and so on. For more information about Eratosthenes' Sieve, check out the web at http://www.math.utah.edu/~alfeld/Eratosthenes.html. One interesting application of large prime numbers is their use in many types of computer security. Look into how prime numbers are used for this purpose.

Investigation & Exploration (Problem #10)

To solve an equation, we usually want to "undo" the operations on both sides of the equation. What operation "undoes" exponentiation? You may not be familiar with the logarithm function, but it is the function that can "undo" a base from its exponent in a problem like $10^x = 100$, where the base is 10. We can rewrite this as log 100 = x. Using a calculator, note that log 100 = 2. That is because $10^2 = 100$.

A property of logarithms is that the log (a^b) = b(log a). If we solve our Warm-Up problem using logs, we have log 20 < log 2^n < log 200 ... notice that we are "taking the log of" each of the three expressions in the inequality. Using our property of logarithms, our inequality becomes log 20 < n(log 2) < log 200. So dividing all three expressions by log 2, we arrive at (log 20)/(log 2) < n < (log 200)/(log 2). Do these calculations on your calculator to see that this interval includes the integers 5, 6 and 7.

Take the equation $2^3 = 8$. We know this to be true. So if we had $2^x = 8$, we would expect x to equal 3. Try solving this equation using logarithms. Start by taking the log of both sides of the equation.