MATHCOUNTS

2004 State Competition Solutions

Are you wondering how we could have possibly thought that a Mathlete® would be able to answer a particular Sprint Round problem without a calculator?

Are you wondering how we could have possibly thought that a Mathlete would be able to answer a particular Target Round problem in less 3 minutes?

Are you wondering how we could have possibly thought that a Mathlete would be able to answer a particular Team Round problem with less that 10 sheets of scratch paper?

The following pages provide solutions to the Sprint, Target and Team Rounds of the 2004 MATHCOUNTS® State Competition. Though these solutions provide creative and concise ways of solving the problems from the competition, there are certainly numerous other solutions that also lead to the correct answer, and may even be more creative or more concise! We encourage you to find numerous solutions and representations for these MATHCOUNTS problems.

Special thanks to volunteer author Mady Bauer for sharing these solutions with us and the rest of the MATHCOUNTS community!

2004 STATE COMPETITION

SPRINT ROUND

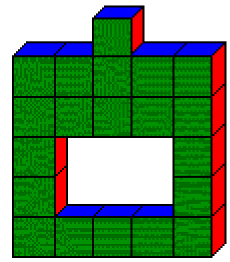
1. Lee uses 2 cups of flour to make 18 cookies. This means he uses 1 cup of flour for every $18 \div 2 = 9$ cookies. Therefore, he can make $9 \times 3 = 27$ cookies with 3 cups of flour.
   27 **Ans.**

2. $2^3 \times 3^x = 72 = 2^3 \times 3^2$
   $x = 2$ **Ans.**

3. The circumference of a circle is 18 cm.
   $C = 2\pi r$
   $18 = 2\pi r$
   $9 = \pi r$
   $r = \frac{9}{\pi}$
   $A = \pi r^2 = \pi \times \left(\frac{9}{\pi}\right)^2$
   $A = \frac{81\pi}{\pi^2} = \frac{81}{\pi}$ **Ans.**

4. The hypotenuse of a right triangle is $\sqrt{97}$.
   One of the legs is 4.
   Let c = the hypotenuse.
   Let b = the leg that is 4.
   $a^2 + b^2 = c^2$
   $a^2 + 4^2 = \left(\sqrt{97}\right)^2$
   $a^2 + 16 = 97$
   $a^2 = 97 - 16 = 81$
   $a = 9$
   The area of the triangle is:
   $\frac{1}{2}ab = \frac{1}{2}(9 \times 4) = \frac{1}{2} \times 36 = 18$
   One square foot is 144 square inches. The area of the triangle is 18 square inches.
   $\frac{18}{144} = \frac{1}{8}$ **Ans.**

5. We need to determine the surface area of this figure.



Start with the front. How many cubes present a surface to the front. On the top level, there is only 1 cube. On the second level there are 5. That's also the same on the third level. On the fourth and fifth level, only 2 cubes present a surface to the front and on the sixth level 5 more present a surface. These are shown in green.
$1 + 5 + 5 + 2 + 2 + 5 = 20$
The same is true for the back of the figure.
Looking down from the top, there are 5 surfaces showing on the top (from the first and second levels). But don't forget that there are 3 showing on the first level as well.
$5 + 3 = 8$
Looking up from the bottom there are 5 from the sixth level and also 3 from the third level, again for a total of 8.
Looking from the right side, there is 1 on the top level, one on each of the other 5 levels, and two inner ones for a total of 8. The same is true looking from the left side.
$20 + 20 + 8 + 8 + 8 + 8 = 72$ **Ans.**

6. Let x = the number of students that one grid line represents. Then there are ($1\frac{1}{2}x$) 6th grade students on the team, $3x$ 7th grade students and ($2\frac{1}{2}x$) 8th grade students.
   There are a total of 14 students on the team.
   $1\frac{1}{2}x + 3x + 2\frac{1}{2}x = 7x$
   $7x = 14$
   $x = 2$
   $1\frac{1}{2} \times 2 = 3$ **Ans.**

7. $80 + 81 + ... + 89 + 90 =$
   $(80 + 90) + (81 + 89) + (82 + 88) +$
   $(83 + 87) + (84 + 86) + 85 =$
   $(170 \times 5) + 85 =$
   $850 + 85 = 935$ **Ans.**

8. $\frac{3^4+3^2}{3^3-3}=\frac{81+9}{27-3}=\frac{90}{24}=\frac{15}{4}$ **Ans.**

9. {a, b, c, d, e} is a set of numbers.
c < d Eq. 1
a < b Eq. 2
d < a Eq. 3
e < d Eq. 4
Eq. 4 says that e < d and
Eq. 1 says that c < d.
Eq. 3 says that a > d and
Eq. 2 says that b > a which is > d.
Therefore, d must be the median.
d **Ans.**

10. The first map displays 8 km as 1 cm. In the new map 5 km is displayed as 1 cm. In the old map the portion of road between the two points is 3.75 cm or 3.75 × 8 = 30 km.
$\frac{30}{5}$ = 6 **Ans.**

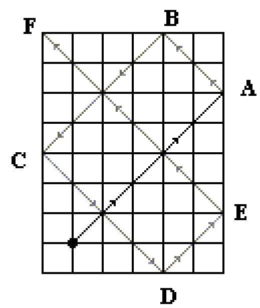
11. An arithmetic sequence is one in which the terms differ by a constant amount. If the 7th term is 30 and the 11th term is 60, then each term differs by:
$\frac{60-30}{11-7}=\frac{30}{4}=7.5$
The 21st term is 10 more terms than the 11th term or 10 × 7.5 = 75 more.
60 + 75 = 135
Even quicker, every four terms increases the value by 30 and every two by 15.
10 = 4 + 4 + 2 terms so,
60 + 30 + 30 + 15 = 135 **Ans.**

12. Let x be the number to add to the numerator and denominator.
$\frac{5+x}{8+x}=0.4=\frac{2}{5}$
5 × (5 + x) = 2 × (8 + x)
25 + 5x = 16 + 2x
3x = -9
x = -3 **Ans.**

13. Let m = the greatest positive three-digit multiple of 7.
Let n = the least positive three-digit multiple of 7.
The greatest three-digit multiple of 7 is 7 × 142 = 994. (Just think 700 + 280 = 980; 980 + 14 = 994). So m = 994.
The least three-digit multiple of 7 is 7 × 15 = 105. (Just think 70 + 28 + 7.) So n = 105.
m + n = 994 + 105 = 1099 **Ans.**

14. Three of the final contestants are female and two are male. If we are to choose twice there are $\frac{5\times4}{2}$ = 10 choices. There are $\frac{3\times2}{2}$ = 3 choices where only females are chosen. $\frac{3}{10}$ **Ans.**

15. A ball is shot from the lower left part of the table along a path of 45 degrees. When it contacts a side, it continues along a path that is a reflection of the path prior to contact.



After the first shot, where the ball hits at A, it continues on to B, C, D, E and F for a total of 5 times that it touches a side before it reaches the corner at point F. 5 **Ans.**

16. A positive multiple of 45 less than 1000 is randomly selected. There are 22 multiples of 45 less than 1000. (Think 450 + 450 + 45 + 45). Of the 22 multiples of 45, there are only two that are two-digit values.
$\frac{2}{22}=\frac{1}{11}$ **Ans.**

17. To determine the units digit of 9^{2004}, look at the powers of 9. 9, 81, 729, 6561... A pattern is forming. Odd powers of 9 end in 9. Even powers of 9 end in 1. Therefore the units digit of 9^{2004} is a 1. To determine the tens digit, look at a few more powers of 9.
…49, …41, …69, …21, …89, …01, …09, …81, etc.
(Notice that we only need to know the last two digits, so we don't have to multiply these large numbers all the way out.) We can see that these values repeat after every 10 powers, so the last tow digits of 92004 will be the same as 94, which is …61 and

the sum of 6 + 1 = 7 **Ans.**

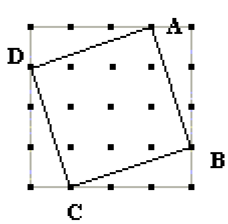
18. 43% of Americans have Type A molecules in their blood. 15% have type B molecules and 46% have neither Type A nor Type B. Therefore, 100% - 46% = 54 % have either Type A or Type B or both.
Let x = the percentage that have both Type A and Type B. Then:
$43 - x$ have only Type A.
$15 - x$ have only Type B.
$43 - x + 15 - x + x = 54$
$58 - 2x + x = 54$
$58 - x = 54$
$x = 4$ **Ans.**

19. The points (x,y) in the table lie on a straight line.

x	y
2	-5
p	-14
p + 2	-17

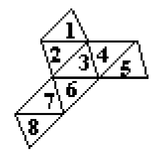
From the last two rows of the table, we can see that when the x-value increases by 2, the y-value decreases by 3. This is the same as saying that as the x-value increases by 1, the y-value decreases by 1.5. From the first row to the second row, there is a decrease in the y-value of 9, which is 1.5×6, so there would be an increase of $1 \times 6 = 6$ from the x-value of 2, so $p = 8$. Now, to go from (2, -5) to (13, q), there is an increase in the x-value of $11 = 1 \times 11$, so there will be a decrease in the y-value of $1.5 \times 11 = 16.5$, so $q = -5 – 16.5 = -21.5$. The value of $p + q$ is $8 + -21.5 = -13.5$ **Ans.**

20. The square ABCD is inside of a 5 by 5 square grid as shown below.



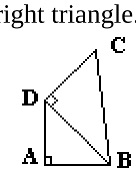
The area of the grid is $4 \times 4 = 16$.
The side of square ABCD is just the hypotenuse of any of the 3x1 right triangles that border the square.
Let x = the hypotenuse.
$x^2 = 3^2 + 1^2 = 9 + 1 = 10$
$x = \sqrt{10}$
The area of square ABCD is
$\left(\sqrt{10}\right)^2 = 10$
The perimeter of square ABCD is
$4 \times \sqrt{10} = 4\sqrt{10}$
The product of the area and the perimeter is:
$10 \times 4\sqrt{10} = 40\sqrt{10}$ **Ans.**

21. The net shown below is folded into a regular octahedron, and each triangle is equilateral.



Once folded, it looks like two square pyramids connected at their square bases. The sides around the top are 1, 2, 3 and 4 and the sides around the bottome are 5, 6, 7 and 8. Side 1 is connected to 2 and 4 on the top and 8 on the bottom. 4 + 8 + 2 = 14 **Ans.**
P.S. Isn't this what scrap paper is for? To cut things out and fold?

22. Each triangle in this figure is an isosceles right triangle.



The length of $\overline{BC}$ is 2 and it is the hypotenuse of triangle CDB. Since this triangle is isosceles the length of $\overline{DB}$ is the same as the length of $\overline{DC}$. Let x be this length.
$x^2 + x^2 = 2^2$
$2x^2 = 4$
$x^2 = 2$
$x = \sqrt{2}$
$\overline{DB}$ is also the hypotenuse of triangle DAB which means that the length of $\overline{AD}$ and the length of $\overline{AB}$ are the same. Call this length y.
$y^2 + y^2 = x^2 = 2$
$2y^2 = 2$
$y^2 = 1$
$y = 1$
The perimeter of quadrilateral ABCD is:
$1 + 2 + \sqrt{2} + 1 = 4 + \sqrt{2}$ **Ans.**

23. Which of these points lies in the region above the line y = 2x + 7 in the coordinate plane?

(3,10), (6, 20), (12, 35), (18, 40) and (20, 50).
First try (3,10). If x = 3, then 2x + 7 = 13. (3,10) is below (3,13).
Next, try (6, 20). If x = 6, then 2x + 7 = 19. (6, 20) is above (6, 19). This qualifies.
Now, try (12, 35). If x = 12, then 2x + 7 = 31. (12, 35) is definitely above (12, 31). This qualifies.
Next, try (18, 40). If x = 18, then 2x + 7 = 43. (18, 40) is below (18, 43).
Finally, try (20, 50). If x = 20, then 2x + 7 = 47. (20, 50) is above (20, 47). This qualifies.
6 + 12 + 20 = 38 **Ans.**

24. x, $x + 2$, $x + 4$, ..., $x + 2n$ form an arithmetic sequence and x is an integer.
$x^3 + (x + 2)^3 + (x + 4)^3 + ... + (x + 2n)^3 = -1197$
and $n > 3$.
Looking for $n > 3$ tells us to be suspicious that there may be more than one solution. In any event, this says that the sum of odd cubes or even cubes will be -1197.
Consider even cubes first. We can look at positive values and then deal with negative, if we have to.
Even cubes are 2^3, 4^3, 6^3, 8^3, 10^3 etc. or
8, 64, 216, 512, 1000
There's no way we can get to 1197 (or using the negative value, -1197). This is because the addition of even numbrs always results in an even integer. Let's look at odd cubes next.
Odd cubes are 1^3, 3^3, 5^3, 7^3, 9^3, etc. or
1, 27, 125, 343, 729
Those numbers look close to 1197. Add them up.
1 + 27 + 125 + 343 + 729 = 1225 = 1197 + 28
1 + 27 = 28
So 125 + 343 + 729 = 1197 or using the negative integers,
-125 + (-343) + (-729) = -1197
Remember, though that $n > 3$. Look at the set of odd cubes again:
-729, -343, -125, -27, -1, 1, 27, 343, 729
If we add the four terms -27, -1, 1, and 27 to the sequence we already know works, we can see that these four terms don't change the end value and we will still get -1197 as the sum. There are 7 terms in the sequence and since the first term is x, the seventh term is $x + 2n$ or $x + 12$.
$2n = 12$
n = 6 **Ans.**

25. The set {5, 8, 10, 18, 19, 28, 30, x} has eight members. The arithmetic mean of the set's numbers is 4.5 less than x.

$$\frac{5+8+10+18+19+28+30+x}{8}+4.5=x$$

$$\frac{118+x}{8}+\frac{36}{8}=x$$

118 + x + 36 = 8x
7x = 154
x = 22 **Ans.**

26. The ratio of 3 times the measure of ∠A to four times the measure of the complement of ∠A to half the measure of the supplement of ∠A is 3:14:4. Huh? Take it slowly.
Let x = ∠A.
Three times the measure of ∠A is 3x.
Four times the measure of the complement of ∠A is 4(90-x).
Half the measure of the supplement of ∠A is $\frac{1}{2}$(180-x).
Put it all together:

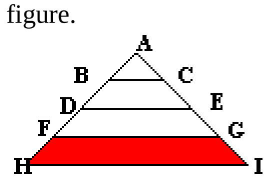
$$\frac{3x}{4(90-x)}=\frac{3}{14}$$

42x = 12(90 - x)
42x = 1080 - 12x
54x = 1080
x = 20
Finish the rest of the ratio.

$$\frac{4(90-x)}{\frac{1}{2}(180-x)}=\frac{14}{4}=\frac{7}{2}$$

8(90 - x) = $\frac{7}{2}$(180 - x)
16(90 - x) = 7(180 - x)
1440 - 16x = 1260 - 7x
180 = 9x
x = 20 again (which proves it and we didn't have to go this far). Therefore, the complement of ∠A is 90 - 20 =
70 **Ans.**

27. Triangle AHI is equilateral as shown in the figure.

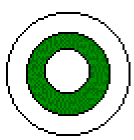


All horizontal segments are parallel and

AB = BD = DF = FH
The parallel segments prove that the four triangles ABC, ADE, AFG and AHI are all similar. We see that AB = $\frac{1}{4}$ AH, and keeping scale factors in mind we can say BC = y, DE = $2y$, FG = $3y$ and HI = $4y$. Similarly, we can let the height of triangle ABC = x, of ADE = $2x$, of AFG = $3x$ and of AHI = $4x$. The area of trapezoid FGHI is then $(1/2)(4y + 3y)x$ and the area of triangle AHI is $(1/2)(4y)(4x)$. The ratio is then

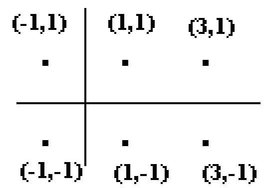
$$\frac{\frac{7xy}{2}}{\frac{16xy}{2}} = \frac{7}{16}$$ **Ans.**

28. The dartboard below has a radius of 6 inches.

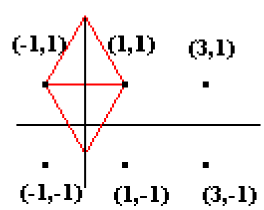


Each of the concentric circles has a radius of two inches less than the next larger circle. This means that the radius of the smallest circle is 2 and the radius of the intermediate circle is 4.
First, find the area of the large circle.
$A = \pi r^2 = \pi \times 6^2 = 36\pi$
To find the area of the shaded region, find the area of the circle with radius of size 4 and subtract from it the area of the circle with radius of size 2.
Area of shaded region is:
$A = \pi \times 4^2 - \pi \times 2^2 = 16\pi - 4\pi = 12\pi$
Thus, the area of the shaded region is $\frac{12\pi}{36\pi} = \frac{1}{3}$ of the area of the large circle.
This means that the probability of a dart hitting the non-shaded region is:
$1 - \frac{1}{3} = \frac{2}{3}$
Of 9 darts hitting the target, we would expect $\frac{2}{3}$ of them, or 6 to hit the target in the non-shaded region. 6 **Ans.**

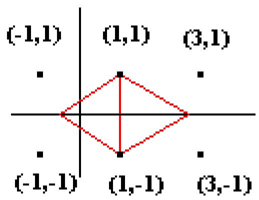
29. The graph shows six points.



We must find all distinct circles of radius 2 which are in the coordinate plane and pass through exactly two of the labeled points.
I was going to try and draw this but it's just too hard with Microsoft Paint so I'll describe it instead.
There are 4 circles using one of the points as the center and two others as part of the circumference of the circle (note that the center point isn't passed through by the circle). So there are:
1. Center (-1,1) passing through (1,1) and (-1, -1).
2. Center (-1,-1) passing through (1,-1) and (-1,1).
3. Center (3, 1) passing through (1,1) and (3,-1).
4. Center (3,-1) passing through (3,1) and (1,-1).
Note that you can't use (1,1) or (1,-1) as the center of a circle because, in each case, you would have the circle passing through **3** of the other points.
The next set is much harder to see. Choose two of the points that are "adjacent", like (-1,1) and (1,1). Draw a line from one point to the next and then construct an equilateral triangle above the two points and draw one below the two points as in this illustration.

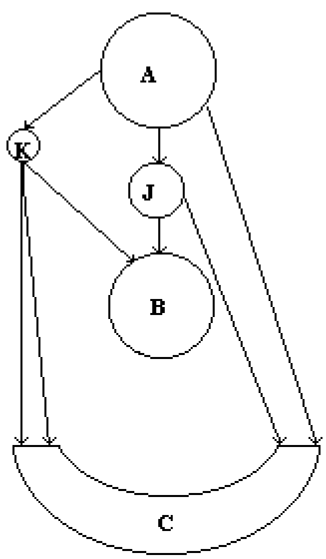


(-1,1) and (1,1) are two points on the circle and the vertex of each of these triangles is the center of a satisfactory circle. You can do this for the following groups of points:
5. and 6. (-1,1) and (1,1) as shown above
7. and 8. (1,1) and (3,1)
9. and 10. (3,1) and (3,-1)
11. and 12. (3,-1) and (1,-1)
13. and 14. (1,-1) and (-1,-1)
15. and 16. (-1,-1) and (-1,1)
But we're not quite done. Don't forget...



17. and 18. (1,1) and (3,-1)
Also notice that the pairs of points (3,1) & (-1,-1) and (-1,1) & (3,-1) are more than 2 units apart and therefore can't contribute to a circle.
Now, that was fun and if you can see that in about 80 seconds, my hat's off to you!!!
18 **Ans.**

30. Regions A, B, C, J and K represent ponds. (Okay, this is really hard to draw!)

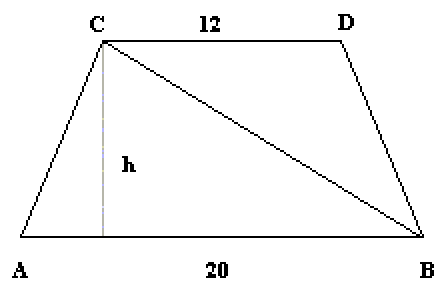


Logs leave pond A and float down flumes (represented by arrows, or sort of arrows...) to eventually end up in pond B or pond C. Logs can only float in the direction of the arrows. We need to determine the probability that a log in pond A will end up in pond B.
How can a log from pond A get to pond B?
$A \rightarrow K \rightarrow B$
$A \rightarrow J \rightarrow B$
The probability of going from A to K to B is:
$$\frac{1}{3}\times\frac{1}{3}=\frac{1}{9}$$
The probability of going from A to J to B is:
$$\frac{1}{3}\times\frac{1}{2}=\frac{1}{6}$$
$$\frac{1}{9}+\frac{1}{6}=\frac{2}{18}+\frac{3}{18}=\frac{5}{18}$$ **Ans.**

TARGET ROUND

1. A circle of radius 20 inches has an area of $\pi r^2 = \pi \times 20^2 = 400\pi$. A circle with diameter 20 inches is actually a circle with radius 10 inches. Its area is $\pi r^2 = \pi \times 10^2 = 100\pi$.
$400\pi - 100\pi = 300\pi$ **Ans.**

2. Quadrilateral ABCD is a trapezoid with $\overline{AB}$ parallel to $\overline{CD}$. AB = 20 and CD = 12.



The area of trapezoid ABCD is
$$\frac{1}{2}h\times(12+20)=\frac{1}{2}h\times 32 = 16h$$
The area of triangle ABC is
$$\frac{1}{2}h\times 20 = 10h$$
$$\frac{10h}{16h}=\frac{10}{16}=\frac{5}{8}$$ **Ans.**

3. A license plate consists of two letters followed by two digits. Neither the digits nor the letters may be repeated. In addition, neither the letter "O" or the digit "0" may be used. Letters must be in alphabetical order and digits must be in increasing order. How many different license plates are there?
Yech! This seems to be hard. Let's look at it a little more closely.
The alphabetic part is separated from the numeric part, i.e., they don't have any dependencies. So let's look at the alphabetic part. If the license starts with "A", then the second letter can be any letter from "B" to "Z" except "O" so there are 24 choices when the license starts with "A". If it starts with "B" there are 23 choices for the second letter (no "A", "B", or "O"). Similarly down to "N" which can have 11 choices. We can't use "O" and "P" can have 10 choices down to "Y" which can only have one. No

license plate can start with Z.
24 + 23 + ... 2 + 1 =

$(24 + 1) \times \frac{24}{2} = 25 \times 12 = 300$ choices

Now let's look at the numbers. If the first number is "1", then the second can be "2" through "9" for a total of 8 choices. If the first number is "2", then the second can be "3" through "9" for a total of 7 choices. Similarly, if the first number is "8", then the second can be "9" for a total of 1 choice. Notice that the first number can't be 9 because the rules say that the numbers must be increasing.

$8 + 7 + ... + 1 = (8 + 1) \times \frac{8}{2} =$

$9 \times 4 = 36$

$300 \times 36 = 10,800$ **Ans.**

4. Mr. Carter pays \$54.30 for water, \$17.00 for gas and \$123.49 for electricity in September. He only has \$120 to pay towards these bills and he distributes the money proportionally to each bill.
First, find out what percentage the water bill is of the total bill. The total of all three bills is \$54.30 + \$17.00 + \$123.49 = \$194.79

The percentage is $\frac{54.3}{194.79} \approx$

0.278761743416 or 27. 8761743416%
Therefore, Mr. Carter will pay this percentage of his money to the water company.
$0.278761743416 \times 120 \approx 33.45140921 \approx$ \$33.45 **Ans.**

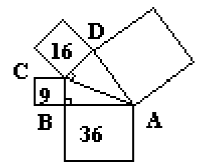
5. What is the largest integer n such that $(1 + 2 + 3 + ... + n)^2 < (1^3 + 2^3 + ... + 7^3)$
$1^3 + 2^3 + ... + 7^3 =$
$1 + 8 + 27 + 64 + 125 + 216 + 343 = 784$
The square root of 784 is 28.
$(1 + 2 + 3 + ... + n) < 28$

$(1 + 2 + 3 + ... + n) = (n+1)\times\frac{n}{2} = \frac{n(n+1)}{2}$

$\frac{n(n+1)}{2} < 28$

$n \times (n + 1) < 56$
So we are looking for the largest value of n such that the product of n and the next larger number is less than 56.
$7 \times 8 = 56$
So, we have to go one less.
$6 \times 7 = 42 < 56$
n = 6 **Ans.**

6. Two right triangles, ABC and ACD, are joined as shown.



Squares are drawn on all four sides. The areas of three of the squares are 9, 16 and 36 square units. The square with side $\overline{AB}$ has a length of 6. The square with side $\overline{BC}$ has a length of 3. Let x = AC.
$6^2 + 3^2 = x^2$
$36 + 9 = 45 = x^2$
The square with side $\overline{CD}$ has a length of 4.
Let y = AD.
$45 + 4^2 = 45 + 16 = 61 = y^2$
The area of the square is just the square of the side. Therefore, the area of the fourth square is 61. **Ans.**

7. What is the sum of all the digits used to write the whole numbers 1 through 110, inclusive? Start with the number 1. It appears every 10 numbers as the ones digit, i.e., 1, 11, 21, etc through 99 for a total of 10 times. 1 appears as the tens digit a total of 10 times (10, 11, ..., 19). 10 + 10 = 20.
From 100 to 110, it appears as the hundreds digit 11 times, as the tens digit, once and as the ones digit once. 20 + 11 + 1 + 1 = 33.
Now look at 2. This is easier than 1. It appears as the ones digit 11 times (2, 12, ..., 102). It appears as the tens digit 10 times.
It does not ever appear as the hundreds digit.
11 + 10 = 21.
The same is true for 3 through 9. Since this is an addition of digits, there is no need to figure out how many 0's there are.
$(33 \times 1) + (21 \times 2) + (21 \times 3) + (21 \times 4) + (21 \times 5) + (21 \times 6) + (21 \times 7) + (21 \times 8) + (21 \times 9) =$
$33 + (21 \times (2 + 3 + 4 + 5 + 6 + 7 + 8 + 9)) =$

$33 + (21 \times 11 \times \frac{8}{2}) = 33 + (21 \times 11 \times 4) =$

$33 + 924 = 957$ **Ans.**

8. In a particular sequence, each term after the first two terms is the sum of the two preceding terms. The first term is 20 and the sixth term is 220. Let x_2 = the second term,

x_3 = the third term, etc.
$x_3 = 20 + x_2$
$x_4 = x_3 + x_2 = 20 + x_2 + x_2 = 20 + 2x_2$
$x_5 = x_4 + x_3 = 20 + 2x_2 + 20 + x_2 = 40 + 3x_2$
$x_6 = 220 = x_5 + x_4 = 40 + 3x_2 + 20 + 2x_2$
$220 = 60 + 5x_2$
$5x_2 = 220 - 60 = 160$
$x_2 = 32$
$x_5 = 40 + 3x_2 = 40 + 3 \times 32 = 40 + 96$
$x_5 = 136$
$x_7 = x_6 + x_5 = 220 + 136 = 356$ **Ans.**

TEAM ROUND

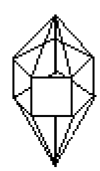
1. $2^{10} \times 4^{20} \times 8^{30} = 2^n$
$2^{10} \times (2 \times 2)^{20} \times (2 \times 2 \times 2)^{30} = 2^n$
$2^{10} \times (2^{20} \times 2^{20}) \times (2^{30} \times 2^{30} \times 2^{30}) = 2^n$
$2^{(10 + 20 + 20 + 30 + 30 + 30)} = 2^n$
$10 + 20 + 20 + 30 + 30 + 30 = n$
$n = 140$ **Ans.**

2. A square and an equilateral triangle have equal perimeters. The area of the triangle is $2\sqrt{3}$. The area of any equilateral triangle is $\frac{1}{2}s \times \frac{\sqrt{3}}{2}s = \frac{\sqrt{3}}{4}s^2$ where s is a side of the triangle.
$2\sqrt{3} = \frac{\sqrt{3}}{4}s^2$
$2 = \frac{1}{4}s^2$
$s^2 = 8$
$s = \sqrt{8}$
Thus, the side of the equilateral triangle is $\sqrt{8}$.
The perimeter of the triangle and square are the same. Let x = the side of the square.
$3\sqrt{8} = 4x$
$x = \frac{3}{4}\sqrt{8}$
Let d = the diagonal of the square.
$d^2 = \left(\frac{3}{4}\sqrt{8}\right)^2 + \left(\frac{3}{4}\sqrt{8}\right)^2$
$d^2 = \left(\frac{9}{16} \times 8\right) + \left(\frac{9}{16} \times 8\right)$
$d^2 = \frac{9}{2} + \frac{9}{2} = 9$
$d = 3$ **Ans.**

3. Coin A is tossed three times and coin B is tossed two times. We are asked to find the probability that more heads are tossed using coin A than using coin B.
Suppose tossing coin A three times gives us 1 head and tossing coin B gives us no heads.
(HTT - TT) or (THT - TT) or (TTH - TT)
The probability is:
$\frac{3}{8}$ for coin A and $\frac{1}{4}$ for coin B, i.e., only 3 of the possible outcomes for coin A and 1 of the possible outcomes for coin B will work.
$\frac{3}{8} \times \frac{1}{4} = \frac{3}{32}$
Suppose tossing coin A three times gives us 2 heads and tossing coin B two times gives us 0 or 1 head.
For coin A, that's HHT, HTH, THH and for coin B that's TT, TH or HT.
$\frac{3}{8} \times \frac{3}{4} = \frac{9}{32}$
Suppose tossing coin A three times gives us three heads. Tossing coin A two times can give us anything and it won't matter because 3 > 2. Therefore, the probability of HHH is $\frac{1}{8}$. Add them up to get the total probability:
$\frac{3}{32} + \frac{9}{32} + \frac{1}{8} = \frac{3}{32} + \frac{9}{32} + \frac{4}{32} = \frac{16}{32} =$
$\frac{1}{2}$ **Ans.**

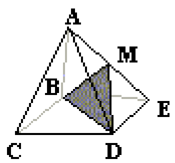
4. A giant earthmover has rubber circular tires 11.5 feet in diameter. How many revolutions does each tire make during a six-mile trip?
6 miles = $5280 \times 6 = 31{,}680$ feet
A revolution is just the circumference of the tire.
$C = \pi d = 11.5\pi \approx 36.12831552$
$\frac{31680}{36.12831552} \approx 876.8745386 \approx 877$ **Ans.**

5. The net with 5 square faces and 10 equilateral triangles is folded into a 15-faced polyhedron as shown below.

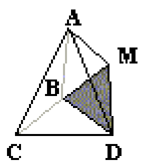


Since there are five squares with four sides each, they contribute 20 sides. There are also 10 triangles with three sides each, which contributes 30 more sides. Notice that every side of a square or triangle hooks up with a side of another square or triangle to form an edge. Therefore, the 50 sides we have will form 50 ÷ 2 = 25 edges. **Ans.**

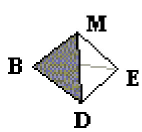
6. A square pyramid ABCDE is sliced along plane BDM into two separate solids.



Point M is between A and E on $\overline{AE}$. The first solid looks as follows:



The number of faces are the triangles BCD, ACD, ABC, BMD, ABM, and ADM for a total of 6 faces. The second figure looks as follows:



The faces are triangles BMD, DEM, DEB, and BEM for a total of 4 faces.
6 + 4 = 10 **Ans.**

7. The set {2, 4, 6, ..., n} contains the positive consecutive even integers from 2 through n. First, let's look at the average of integers in a set like this. The average of the set {2} is 2. The average of the set {2, 4} is 3. The average of the set {2, 4, 6} is 4. There is a pattern here. The average of the set

{2, 4, 6, ..., n} is $\frac{n}{2}+1$.

Now, find the set of positive consecutive even integers whose average is 28.

$\frac{n}{2}+1 = 28$
n + 2 = 56
n = 54
So the average of {2, 4, 6, ..., 52, 54} is 28.
Look at one less or {2, 4, 6, ..., 50, 52}.
The average is $\frac{52}{2} + 1 = 26 + 1 = 27$.
The total sum is $27 \times 26 = 702$.
If we subtract some number from 702 and divide by 25, that result would have to be 28.
$\frac{702 - x}{25} = 28$
702 - x = 700
x = 2
Therefore, we can subtract 2 from the set, i.e., {4, 6, 8, ..., 50, 52) for an average of 28.
What about n = 50?
{2, 4, 6, ..., 48, 50} has an average of 26
$25 \times 26 = 650$
$\frac{650 - x}{24} = 28$
650 - x = 672
x = -22
That's not part of the set. n = 52 **Ans.**

8. The first and second terms of a sequence are A and B, respectively. If x and y are any two consecutive terms of the sequence, with x coming before y, the next term is 2x - 3y. The 5th term of the sequence is written in the form mA + nB. To determine what m and n are we must first determine what the third, fourth and fifth terms are. Let the terms be x_1, x_2, x_3, x_4, and x_5.
$x_1 = A$
$x_2 = B$
$x_3 = 2x_1 - 3x_2 = 2A - 3B$
$x_4 = 2x_2 - 3x_3 = 2B - 3(2A - 3B)$
$= 2B - 6A + 9B = -6A + 11B$
$x_5 = 2x_3 - 3x_4 = 2(2A - 3B) - 3(-6A + 11B)$
$= 4A - 6B + 18A - 33B = 22A - 39B$
m = 22
n = -39
m - n = 22 - (-39) = 22 + 39 = 61 **Ans.**

9. In each blank below a single digit is inserted such that the following six three-digit numbers, in this order, from an arithmetic sequence:
1_ _, _ _ 9, 2 _ 2, _ 6 _, 2 _ _, _ 3 _
Oh! These are so much fun! Especially when there's so little time to do them in!!! Start by looking at the third and fifth numbers. They each have a "2" as the hundreds digit. Since this is an arithmetic sequence, the fourth number has a 2 in the hundreds digit.
1_ _, _ _ 9, 2 _ 2, 2 6 _, 2 _ _, _ 3 _
Looking at the second and third numbers it

appears that the ones digit differs by 3. This implies that the ones digit of the fourth term will be 3 larger than that of the third term 2 + 3 = 5

1_ _, _ _ 9, 2 _ 2, 2 6 5, 2 _ _, _ 3 _

Similarly, fill in the ones digits for the first, fifth and sixth terms.

1_ 6, _ _ 9, 2 _ 2, 2 6 5, 2 _ 8, _ 3 1

Looking at the fourth and sixth terms, 65 is greater than 31. And the third through fifth terms are in the 200's. So it is safe to assume that the sixth term is in the 300's.

1_ 6, _ _ 9, 2 _ 2, 2 6 5, 2 _ 8, 3 3 1

Going back to the fourth through sixth terms, we can take the difference.

331 - 265 = 66

This means that each term is 33 larger than the previous term.

1 6 6, 1 9 9, 2 3 2, 2 6 5, 2 9 8, 3 3 1

Therefore the seventh term is 331 + 33 = 364 **Ans.**

10. Four CD's are rap. Five are country music and 3 are heavy metal. Five of the CD's are randomly selected to purchase. One of each type of CD must be in the five that are purchased.

There are two things that need to be determined. The first thing is to list the possible scenarios under which at least one of each CD can be picked and determine the probability of that happening. The second thing is to determine how many of each type of scenario exists.

The scenarios under which at least one of each CD is picked are:

1. 1 Country, 3 Rap, 1 Heavy Metal
2. 3 Country, 1 Rap, 1 Heavy Metal
3. 1 Country, 1 Rap, 3 Heavy Metal
4. 2 Country, 2 Rap, 1 Heavy Metal
5. 1 Country, 2 Rap, 2 Heavy Metal
6. 2 Country, 1 Rap, 2 Heavy Metal

Next, we figure out how many ways each of these scenarios can occur.

1. ${}_5C_1 \times {}_4C_3 \times {}_3C_1 = 5 \times 4 \times 3 = 60$ ways (1C, 3R, 1H)
2. ${}_5C_3 \times {}_4C_1 \times {}_3C_1 = 10 \times 4 \times 3 = 120$ ways (3C, 1R, 1H)
3. ${}_5C_1 \times {}_4C_1 \times {}_3C_3 = 5 \times 4 \times 1 = 20$ ways (1C, 1R, 3H)
4. ${}_5C_2 \times {}_4C_2 \times {}_3C_1 = 10 \times 6 \times 3 = 180$ ways (2C, 2R, 1H)
5. ${}_5C_1 \times {}_4C_2 \times {}_3C_2 = 5 \times 6 \times 3 = 90$ ways (1C, 2R, 2H)
6. ${}_5C_2 \times {}_4C_1 \times {}_3C_2 = 10 \times 4 \times 3 = 120$ ways (2C, 1R, 2H)

This is a total of 60 + 120 + 20 + 180 + 90 + 120 = 590 ways five CDs can be chosen while guaranteeing that at least on of each type is selected. There are ${}_{12}C_5 = 792$ ways to just pick five of the 12 CDs without any restrictions. So the probability of success is

$590/792 = \frac{295}{396}$ **Ans.**

P.S. I'm tired....