# 2006 MATHCOUNTS STATE COMPETITION SOLUTIONS

There may be a more elegant way, but the following is one example of how to solve each of the competition problems.

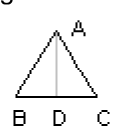
## SPRINT ROUND

1. The Poe M.S. students are assigned to teams of no more than eight students and each student is to be on exactly one team. If there are 128 students, dividing the number of students by 8 gives us the minimum number of teams that can be formed.
$\frac{128}{8}$ = 16 **Ans.**

2. The graph shows that Carla studied 0 hours on 2 days, 1 hour on 5 days, 2 hours on 8 days, 3 hours on 10 days, 4 hours on 3 days and 5 hours on 3 days.
10 + 3 + 3 = 16 **Ans.**

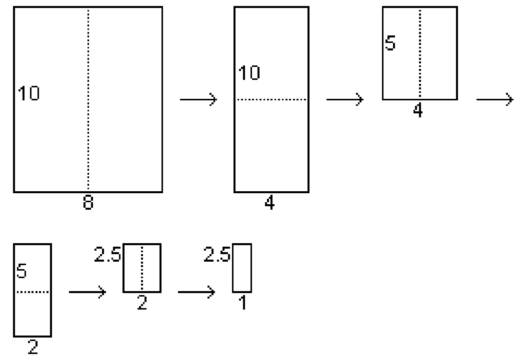
3. Sasha has $3.20 in coins. She has the same number of quarters and nickels. To find the greatest number of quarters she can have, first compute the largest multiple of 30 less than $3.20 (25¢ + 5¢ = 30¢)
30 × 10 = 300
Thus, 10 quarters and 10 nickels gives us $3.00. The rest of the $.20 that is left over must be in other coins.
10 **Ans.**

4. In triangle ABC, point D is on segment BC, the measure of angle BAC is 40º, and triangle ABD is a reflection of triangle ACD over segment AD.




The reflection information tells us that angle BAD = angle DAC = 20º. Both angle BDA and ACD are right angles. Thus angle B must be:
180 – (20 + 90) =
180 – 110 = 70 **Ans.**

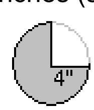
5. Lizzie folded a piece of 8-inch by 10-inch paper in half again and again until she ended up with a folded piece that measured 1 inch by 2.5 inches. Let's draw this.




You can see that it takes 5 folds. To do it without drawing the picture, start with 10x8. Divide the 8 by 2 and get a 10x4. That's 1 fold. Divide the 10 by 2 and get 5x4. That's 2 folds. Divide 4 by 2 and get 5x2. That's 3 folds. Divide 5 by 2 and get 2.5x2. That's 4 folds. Divide 2 by 2 and get 2.5x1 and that's 5 folds. 5 **Ans.**

6. Allen's heart beat 100,000 times in 1 day. This means it beat
$\frac{1000000}{24}$ times in 1 hour or
$\frac{1000000}{24} \times 6 = \frac{1000000}{4}$ =
25000 times in 6 hours.
25,000 **Ans.**

7. A paper cone is to be made from a three-quarter circle having radius 4 inches (shaded).




The length of the arc on the discarded quarter-circle (outline in grey) is just one-quarter of the circumference of the circle.
C = 2π4
$\frac{1}{4}$C = $\frac{1}{4}$ × 2 × π × 4 =
2π **Ans.**

8. The letters of the alphabet are given numeric values based on two conditions:
- Only the values -2, -1, 0, 1 and 2 are used.
- Starting with A and going through Z, a numeric value is assigned to each letter according to the following pattern:
1, 2, 1, 0, -1, -2, -1, 0, 1, 2, 1, 0, -1, -2, -1, 0, …
We are asked to find the sum of the numeric values of the letters in the word “numeric”.
The series of digits given above repeat every 8 letters. (i.e., 1, 2, 1, 0, -1, -2, -1, 0). We have only to figure out the modulus for each letter in the word numeric (with modulus of 0 corresponding to the eighth digit in the series).
The letter ‘n’ is the 14th letter of the alphabet.
14 mod 8 = 6
The 6th item in the series is -2.
The letter ‘u’ is the 21st letter of the alphabet.
21 mod 8 = 5
The 5th item in the series is -1.
The letter ‘m’ is the 13th letter of the alphabet.
13 mod 8 = 5
The 5th item in the series is -1.
The letter ‘e’ is the 5th letter of the alphabet.
5 mod 8 = 5
The 5th item in the series is -1.
The letter ‘r’ is the 18th letter of the alphabet.
18 mod 8 = 2
The 2nd item in the series is 2.
The letter ‘i’ is the 9th letter of the alphabet.
9 mod 8 = 1
The first item in the series is 1.
The letter ‘c’ is the third letter of the alphabet.
3 mod 8 = 3
The third item in the series is 1.
-2 + -1 + -1 + -1 + 2 + 1 + 1 =
-1 **Ans.**

Alternatively, we could have made a chart:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 1 | 0 | -1 | -2 | -1 | 0 |
| A | B | **C** | D | **E** | F | G | H |
| **I** | J | K | L | **M** | **N** | O | P |
| Q | **R** | S | T | **U** | V | W | X |
| Y | Z | | | | | | |

And now we can see that the sum of the values is 1 + 2 + 1 + -1 + -1 + -1 + -2 = -1.

9. We are asked to find the arithmetic mean of the areas of all non-congruent rectangles with integer side lengths and perimeter 8 units.
p = 2(l + w) = 8
l + w = 4
The requirement for non-congruent rectangles means we count l = 1, w = 3 and w = 3, l = 1 as the same rectangle for this problem. The only other possibility is l = w = 2.
$A_1 = 1 \times 3 = 3$
$A_2 = 2 \times 2 = 4$
$\frac{3+4}{2} = 3.5$ **Ans.**

10. What is the positive value of the expression $\sqrt{x^3 - 2^y}$ when x = 5 and y = 2? Substituting into the expression gives us:
$\sqrt{5^3 - 2^2} = \sqrt{125 - 4} = \sqrt{121}$
= 11 **Ans.**

11. When the two-digit integer MM, with equal digits, is multiplied by the one-digit integer M, the result is the three-digit integer NPM. We are asked to find the greatest possible value of NPM. Note that M × M has an M in the one’s column. Which single digits have squares that satisfy this requirement?
1 × 1 = 1
5 × 5 = 25
6 × 6 = 36
Clearly 6 is the biggest of these numbers.
66 × 6 = 396 **Ans.**

12. A 20-gallon container is filled halfway with a mixture that is 90% vinegar and 10% water. We are asked to find how many gallons of water it takes to create a mixture

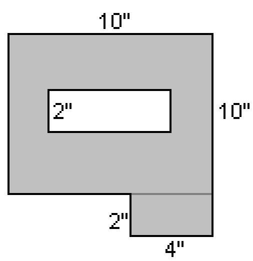
that is 60% vinegar and 40% water.

Since the container is only filled to halfway in the beginning, this means only 10 gallons are in it and 90% of that is 9 gallons of vinegar and 1 gallon of water.
Let x = the number of gallons of water it would take to make the mixture 60% vinegar and 40% water.

$$\frac{9}{10+x} = \frac{6}{10} = \frac{3}{5}$$

5 × 9 = 3 × (10 + x)
45 = 30 + 3x
15 = 3x
x = 5 **<u>Ans.</u>**

13. The area of the shaded region is 78 square inches. All angles are right angles and all measurements are given in inches.




We are asked to find the perimeter of the non-shaded region. We know its height, which is 2 inches but don't know its width.
Let x = the width of the non-shaded area. The region, itself, is made up of a 10x8 rectangle and a 2x4 rectangle. The total area when not dealing with shadings is:
(10 × 8) + (2 × 4) = 80 + 8 = 88
However, we are told that the area of the non-shaded area is 78.
88 – 78 = 10
Thus, the area of the non-shaded portion is 10 inches.
2x = 10
x = 5
The length is 5 so the perimeter is:
2 + 5 + 2 + 5 = 14 **<u>Ans.</u>**

14. A robot takes 2.5 hours to travel 1 kilometer. The robot takes 90 seconds to travel a hallway and we are asked to determine the length of the hallway.
2.5 hours = 2.5 × 60 = 150 minutes = 150 × 60 = 9000 seconds.
1 kilometer is 1000 meters.
Thus, the robot takes 9000 seconds to travel 1000 meters.
Let x = the length of the hallway.

$$\frac{9000}{1000} = \frac{90}{x}$$

9000x = 90000
x = 10 **<u>Ans.</u>**

15. A collection of 5 positive integers has mean 4.4, unique mode 3 and median 4. What is the new median if an 8 is added to the collection?
The 5 integers have the mean 4.4 which means that 5 × 4.4 = 22 is the sum of the 5 integers. If an 8 is added the sum will be 30.
The median is 4 which says that 2 of the values are greater than 4. Why not possibly equal to 4? Because there is a unique mode of 3. That means there are two 3's. This would give us: 3, 3, 4, x, y. Neither x nor y can be 4 because then we would have 2 4's which wouldn't make the mode unique. We can't have 3 3's because then 3 would be the median. We can't have 3 4's because then 4 would be the mode.
22 = 3 + 3 + 4 + x + y
x + y = 12; x > 4; y > x
x = 5; y = 7; possible
x = 6; y = 6 (no because then 3 is not a unique mode)
Thus we actually have:
3, 3, 4, 5, 7, 8
The new median is the average of the middle 2 values.

$$\frac{4+5}{2} = \frac{9}{2} = 4.5$$ **<u>Ans.</u>**

16. A strip of paper consists of eight squares as shown.

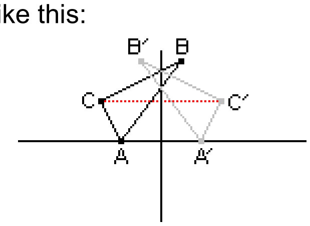
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|

The strip is folded in half so that the right-most square (8) lands face-down and on top of the left-most square (1). We can represent this

as:
8 7 6 5
1 2 3 4
Then the new right-most square is folded over on top of the new left-most square. This gives us:

4 3
5 6
8 7
1 2
And then it's folded again giving is:
2
7
6
3
4
5
8
1
The number on top of the stack is 2. **Ans.**
Note: The easiest way to do this – Did you have scrap paper? Make the strip! But otherwise, what helps is to realize that the bottom right numbers appear in the opposite order on top after the fold and the bottom left numbers appear in the correct order still on the bottom.

17. Triangle ABC with vertices A(-2,0), B(1,4) and C(-3,2) is reflected over the y-axis to form triangle A'B'C'. We are asked to determine the length of a segment drawn from C to C'. Drawing the two triangles looks like this:




Since triangle ABC is reflected over the y axis, the new points A', B' and C' have the same y coordinates as A, B, and C, respectively, but the x coordinates are multiplied by -1. So the new points are A'(2,0), B'(-1,4) and C'(3,2). The distance between C and C' is just
3 - -3 = 6 **Ans.**

18. Points A and B are located at $\frac{1}{8}$ and $\frac{1}{4}$, respectively, on a number line. We are asked to find the sum of the coordinates of the two points that trisect segment AB. The length of segment AB is:
$\frac{1}{4} - \frac{1}{8} = \frac{1}{8}$
Trisecting the segment means breaking it into thirds.
$\frac{1}{8} \times \frac{1}{3} = \frac{1}{24}$
Thus, each of the points between $\frac{1}{8}$ and $\frac{1}{4}$ are $\frac{1}{24}$ apart. Thus the two points that trisect segment AB are:
$\frac{1}{8} + \frac{1}{24} = \frac{3}{24} + \frac{1}{24} = \frac{4}{24}$ and
$\frac{1}{8} + \frac{2}{24} = \frac{3}{24} + \frac{2}{24} = \frac{5}{24}$
$\frac{4}{24} + \frac{5}{24} = \frac{9}{24} = \frac{3}{8}$ **Ans.**

19. There are twice as many 11th-grade students as there are 9th and 10th-grade students combined in the statistics class. There are twice as many 12th-grade students as 9th-grade students. The number of 11th-grade students is ten times the number of 12th-grade students. The class has 32 students and we are asked to find how many 12th-grade students are in the class.
Let a = the number of 9th-grade students.
Let b = the number of 10th-grade students.
Let c = the number of 11th-grade students.
Let d = the number of 12th-grade students.
From the first sentence:
c = 2(a + b)

From the second sentence:
d = 2a
From the third sentence:
c = 10d
And since there are 32 students in the class:
a + b + c + d = 32
Let's see if we can turn this equation with four variables into one with only one variable: a.
c = 10d = 10 × 2a = 20a
c = 2(a + b) = 20a
a + b = 10a
b = 9a
a + 9a + 20a + 2a = 32
32a = 32
a = 1
d = 2a = 2 **Ans.**

This solution certainly works, but notice we were told the number of 11th graders is 10 times the number of 12th graders. With only 30 students in the class, there is either one 12th grader and ten 11th graders or two 12th graders and 20 11th graders. (Notice, three 12th graders leads to 30 11th graders and already we're up to too many kids.). Based on the statement that there were twice as many 12th graders as 9th graders, there must have been two 12th graders.

20. There is a point (x,y) in the first quadrant, and on the line 3x – 5y = 12, for which the x-coordinate is three times the y-coordinate.
This means that:
3(3y) – 5y = 12
9y – 5y = 12
4y = 12
y = 3
x = 3y = 9
9 + 3 = 12 **Ans.**

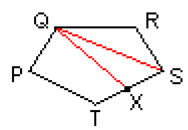
21. Four red candies and three green candies can be combined to make many different "flavors." Flavors are different if the percent red is different. We are asked to determine how many different flavors are possible if flavors can be made by using some or all of the seven candies. Start with only one candy. We have 1 red (100% red) and 1 green (0% red) for a total of 2 flavors. Now try 2 different candies. We have 2 red (100% red), 2 green (0% red) and 1 red/1 green (50% red). The first two flavors are already covered by 1 red and 1 green so we have just 1 new flavor (50% red). Now try 3 candies. We have 3 red (100% red), 2 red/1 green ($66\frac{2}{3}$ % red), 1 red/2 green ($33\frac{1}{3}$ % red), 3 green (0 % red) for 2 new flavors. Try 4 candies: 4 red (100% red), 3 red/1 green (75% red), 2 red/2 green (50% red), 1 red/3 green (25% red) for a total of 2 new flavors. Try 5 candies: 4 red/1 green (80% red), 3 red/2 green (60% red), 2 red/3 green (40% red) for a total of 3 new flavors. For 6 candies we have 4 red/2 greens ($66\frac{2}{3}$ % red), 3 red/3 greens (50% red). We've seen both of these before. Finally there is 4 red/3 green ($57\frac{1}{7}$ % red) which is a new flavor.
2 + 1 + 2 + 2 + 3 + 0 + 1 = 11 **Ans.**

22. The sum of 5 different positive integers is 320. The sum of the greatest three integers in this set is 283. The sum of the greatest and least integers is 119. If x is the greatest integer in the set, what is the positive difference between the greatest possible value and least possible value for x?
Let a, b, c, and d be the other 4 integers in increasing value.
a + b + c + d + x = 320
c + d + x = 283
a + x = 119
a + b = 320 – 283 = 37
Start with the smallest possible value for a. If a = 1, then b = 36.
1 + x = 119
x = 118
Does this work?
1 + 36 + c + d + 119 = 320

156 + c + d = 320
c + d = 164
There are plenty of values for c and d that would satisfy the requirements. How large can a get? Well, it can't be larger than b, so the largest it can be is $\frac{1}{2} \times 37 = 18$ (b would be 19 since it has to be larger). Does 18 work?
18 + x = 119
x = 101
c + d = 283 – 101 – 172
172 permits multiple values of c and d that work. Thus, the range of x is from 101 to 118.
118 – 101 = 17 **Ans.**

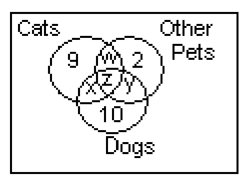
23. In regular pentagon PQRST, X is the midpoint of segment ST. We are asked to find the measure of angle XQS.




First determine the number of degrees in each angle in the pentagon. This follows the formula: 180 × (n – 2) where n is the number of sides in the regular graphic.
180 × (5 – 2) = 180 × 3 = 540
The interior angles of the pentagon sum up to 540º. Each interior angle of the pentagon is:
$\frac{540}{5}$ =108 degrees
Triangle RQS is an isosceles triangle with angle R = 108º. Thus angle SQR and RSQ are both
$\frac{180 - 108}{2} = \frac{72}{2}$ = 36º each.
Angle RST = 108
Angle XSQ = Angle RST – Angle RSQ = 108 – 36 = 72
Since X is the midpoint of segment ST and we have a regular pentagon, QX intersects ST in a 90º angle and QXS = 90
Therefore, angle XQS =
180 – (90 + 72) =
180 – 162 = 18 **Ans.**

24. In the game "Cover It Up" two standard six-faced dice are rolled and their sum determined. The player can then cover that number on that game board or any two numbers that have that sum. We are asked to find the probability of being able to cover the 9 on the first roll of the two dice. To cover the 9 we must have a roll of the dice that is greater than or equal to 9. What values are these? Clearly 9, 10, 11 and 12. 9 can be rolled as (4,5), (5,4), (6,3) and (3,6). 10 can be rolled as (6, 4), (4, 6) and (5, 5). 11 can be rolled as (6, 5) and (5, 6). Finally, 12 can be gotten as (6, 6). This is a total of 4 + 3 + 2 + 1 = 10 possible throws. There are 6 × 6 = 36 possible throws.
$\frac{10}{36} = \frac{5}{18}$ **Ans.**

25. Jeremy made a Venn diagram showing the number of students in his class who own types of pets.




There are 32 students in Jeremy's class. Half of the students have a dog, $\frac{3}{8}$ have a cat, 6 have some other pet and five have no pet at all. We are asked to determine how many students have a cat, a dog and some other pet, i.e., z.
9 + x + w + z = $\frac{3}{8} \times 32 = 12$ (Eq. 1)
2 + y + z + w = 6 (Eq. 2)
10 + x + z + y = 16 (Eq. 3)
9 + 10 + 2 + x + z + y + w = 32 – 5 (Eq. 4)
Simplifying:
x + w + z = 3 (from Eq. 1)
y + w + z = 4 (from Eq. 2)
x + y + z = 6 (from Eq. 3)

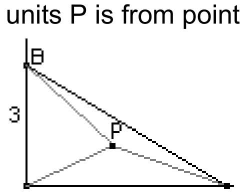
$x + z + y + w = 6$ (from Eq. 4)
Subtracting the last 2:
$w = 0$
$y + z = 4$
$x + z = 3$
$x + z + y = 6$
$x + 4 = 6$
$x = 2$
$2 + z = 3$
$z = 1$ **Ans.**

26. The digits of a four-digit positive integer add up to 14. The sum of the two middle digits is nine, and the thousands digit minus the units digit is one.
Let abcd be the four-digit positive integer. Then:
$a + b + c + d = 14$
$b + c = 9$
$a – d = 1$
And, by the way, the integer is divisible by 11.
$a + 9 + d = 14$
$a + d = 5$
$a – d = 1$
$2a = 6$
$a = 3$
$d = 2$
Thus, the integer is 3bc2.
The following numbers are candidates:
3182, 3812, 3272, 3722, 3362, 3632, 3452, 3542.
Some close numbers to these that are divisible by 11 are 3190, 3300, 3410, 3520, 3630, 3740, 3850.
3182 is only 8 less than 3190. No.
3812 is 72 more than 3740. No.
3272 is 82 more than 3190. No.
3722 is 18 less than 3740. No.
3362 is 62 more than 3300. No.
3632 is 2 more than 3630. No.
3452 is 42 more than 3410. No.
3542 is 22 more than 3410. Yes.
3542 **Ans.**

27. How many positive integers divisible by 4 can be formed using the digits 1, 2, 3 and 4, each at most once for each integer?
Single digits: 4
Double digits: 12, 24, 32
For any integers larger than 2 digits, if the digits are divisible by 4, then the entire integer is divisible by 4.
Triple digits: any that end in 12, 24 and 32. 312, 412, 124, 324, 132, 432 for 6 (that's two different values for the hundreds place and 3 potential double digit endings).
Four digits: Again, they just have to end in 12, 24 or 32.
There are two digits available for each of the double digits (i.e., 3412 and 4312) so $2 \times 3 = 6$
$1 + 3 + 6 + 6 = 16$ **Ans.**

28. Derek's phone number is 336-7624. It has the property that the three-digit prefix, 336, equals the product of the last four digits. We are asked to determine how many seven-digit phone numbers beginning with 336 have this property. What are the prime factors of 336 (include 1)?
7, 3, 2, 2, 2, 2, 1
Notice, the 7 will have to be one of the digits, since it can't be multiplied by any of the other factors and remain a single digit.
The four digits can be:
7, 6, 2, 4
7, 6, 8, 1
7 3, 4, 4
7 3, 8, 2
There are 4! different combinations of 7624 or 24. The same holds for 7681 and 7382. For 7344 there are only 12.
$24 + 24 + 24 + 12 = 84$ **Ans.**

29. Triangle ABC has vertices A(0,0), B(0,3) and C(5,0). A point P inside the triangle is $\sqrt{10}$ units from point A and $\sqrt{13}$ units from point B. We are asked to determine how many units P is from point C.



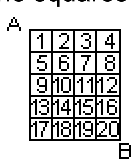

Let the coordinates of point P be (x,y). Then
$x^2 + y^2 = \sqrt{10}^2 = 10$

$x^2 + (3 - y)^2 = \sqrt{13}^2 = 13$

Subtracting the two equations we get:
$(3 - y)^2 - y^2 = 13 - 10 = 3$
$9 - 6y + y^2 - y^2 = 3$
$9 - 6y = 3$
$6y = 6$
$y = 1$
$x^2 + 1 = 10$
$x^2 = 9$
$x = 3$
Finally to determine how far point P is from point C, realize that line PC is the hypotenuse of a triangle with lengths (5 – 3) and 1.
Let z = the length of line PC.
$(5 - 3)^2 + 1^2 = PC^2$
$4 + 1 = 5 = PC^2$
PC = $\sqrt{5}$ **Ans.**

30. Matt arranges four identical dotless dominoes on the 5 by 4 grid so that a path is formed from the upper left-hand corner A to the lower right-hand corner B. We are asked to determine the number of paths from point A to point B where consecutive dominoes must touch at their sides and not just their corners and each domino covers exactly two of the unit squares. Let's number each of the squares as follows:



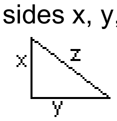

First, notice that we must always make progress toward the bottom corner. We don't have enough dominoes to backtrack or waste moves. Represent a domino on the grid by the numbers of the two squares the domino occupies, enclosed by brackets. So we can start off at corner A as either {1,2} or {1,5}
Using an arrow to indicate that we can get from the left side of the arrow to the positions on the right side of the arrow we get:
{1,2}→{6,10}, {3,7}, {3,4}, {6,7}
{6,10}→{14,18}, {11,12}, {11,15}, {14,15}
{14,18) →{19,20}
{11,12}→{16,20}
{11,15}→{19,20},{16,20}
{14,15}→{19,20},{16,20}
Thus, from {1,2} we can go to {6,10} and from there we can go to {14,18} and {19,20} or {11,12} and {16,20} or {11,15} and {19,20} or {11,15} and {16,20} or {14,15} and {19,20} or {14,15} and {16,20} for a total of 6 paths. {6,10} to the corner has 6 paths. Continuing…
{3.7}→{8,12}, {11,12}, {11,15}
{8,12) →{16,20}
{8,12} is one path and we see from before that {11,12} is one path and {11,15} is two paths so {3,7} has a total of 4 paths to point B.
{3,4}→{8,12} or one path
{6,7}→{11,15}, {8,12}, {11,12} so {6,7} has a total of 4 paths to point B.
Thus from {1,2}, we have a total of 6 + 4 + 1 + 4 = 15 paths to point B.
{1,5}→{6,10}, {6,7}, {9,13}, {9,10}
{9,13}→{17,18}, {14,18}, {14,15}
{17,18}→{19,20}
{9,13} has a total of 4 paths to point B.
{9,10}→{14,18}, {14,15}, {11,15}, {11,12} for a total of 6 paths.
From {1,5} we can get to point B in a total of 6 + 4 + 4 + 6 = 20 paths.
20 + 15 = 35 **Ans.**

Alternately, we know half of the first domino will have to be in the 1 spot. There are seven more half-dominoes to place and we need to make three moves to the right and four moves down – in any order – to get to the bottom corner. There are 7C3 = **35** ways to choose the order of the right moves. For instance, the path described before as {1,2}→{3,7}→{11,12}→{16,20} would be described as 1RRDDRDD in this solution. In this path, the Rs are in the first, second and fifth positions of the seven moves up for grabs.

## TARGET ROUND

1. A magazine subscription saves 75.6% over the cover price for 17 issues versus buying them separately at the cover price. The 17-issue subscription costs $10.99. Determine what the average cover price of each magazine is when bought at a newsstand.
Let x = the cost of buying 17 magazines at the newsstand.
$10.99 = (1 - 0.756) \times x = 0.244x$
$x = \frac{10.99}{0.244} \approx 45.0409836$
Divide this value by 17 to determine the average price of the magazine at a newsstand.
$\frac{45.0409836}{17} \approx 2.6494696$
$\approx 2.65$ **Ans.**

2. A triangle has two sides of length 6 cm and 8 cm and it also has a right angle. What is the shortest possible length of the remaining side of the triangle?
Draw a right triangle and label the sides x, y, and z.




Let x = 6 and y = 8. This is just a copy of a 3, 4, 5 right triangle and we know that z = 10.
Now let x = 6 and z = 8.
$6^2 + y^2 = 8^2$
$36 + y^2 = 64$
$y^2 = 64 - 36 = 28$
$y = \sqrt{28} \approx 5.29150 \approx 5.29$
Now, let y = 8 and z = 6.
$x^2 + 8^2 = 6^2$ but then the hypotenuse is shorter than the leg and we can't have that.
So we have a choice between 10 and 5.29 for the shortest possible length. Hmmm… 5.29 **Ans.**

3. A set of four consecutive integers has a sum of 22. If each integer of the set is increased by 2 and then multiplied by 20, what is the sum of the new set of integers?
Let x = the first integer. Then:
$x + x + 1 + x + 2 + x + 3 = 22$
$4x + 6 = 22$
$4x = 16$
$x = 4$
The four consecutive integers are 4, 5, 6 and 7. Now add 2 to each and then multiply by 20.
$((4+2)\times 20) + ((5+2)\times 20) + ((6+2)\times 20) + ((7+2)\times 20) =$
$(6 \times 20) + (7 \times 20) + (8 \times 20) + (9 \times 20) =$
$(6 + 7 + 8 + 9) \times 20 =$
$30 \times 20 = 600$ **Ans.**
Note that we can do this in a slightly simpler way. We're going to add 2 to each value, which is the same as adding 8 to 22 which is 30. Take that and multiply by 20 and you're done. You don't have to figure out x at all! But I like using the equation writer…

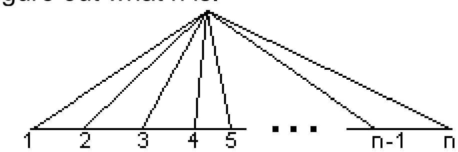
4. The triple bar graph shows that 73% of Californians surveyed, 44% of Alaskans and 14% of Texans call soft drinks Soda. It also shows that 5% of Californians, 42% of Alaskans and 2% of Texans call it Pop. Finally, it shows that 17% of Californians, 7% of Alaskans and 80% of Texans call it Coke®. 8300 people were surveyed in California, 300 in Alaska and 7250 in Texas. We need to determine how many more people surveyed call soft drinks Coke® than Soda.
Let's start with Coke®.
$(0.17 \times 8300) + (0.07 \times 300) + (0.8 \times 7250) = 1411 + 21 + 5800 = 7232$ people surveyed called soft drinks Coke®. Now let's look at Soda.
$(0.73 \times 8300) + (0.44 \times 300) + (0.14 \times 7250) = 6059 + 132 + 1015 = 7206$ people surveyed called soft drinks Soda.
$7232 - 7206 = 26$ **Ans.**

5. A complete lap around a circular track is 400 meters. Jun runs clockwise at 3 meters per second and Quan runs counterclockwise at

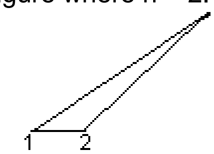
5 meters per second. They start running at the starting line. When they meet for the sixth time after starting, they stop and both walk back together along the track to the starting line. We are asked to determine the shortest distance they could walk back on the track together.
Where do they meet the first time? Clearly somewhere not at the start point and neither of them will have completed a lap but the sum of the distance each has traveled will be an entire lap.
Let x = the amount of time they run until they meet.
$3x + 5x = 400$
$8x = 400$ (This makes sense since each of them running in the opposite directions separates them at a rate of 8 meters per second.)
$x = 50$
Thus, they meet in 50 seconds. So the sixth time they meet will be in 6(50) = 300 seconds. Quan will have run (5)(300) = 1500 meters, which is 100 meters shy of finishing his fourth lap. So they will meet here and can either walk the 100 meters to complete his lap or the 300 meters that he'd already run. The shortest way back is to walk counterclockwise for 100 meters.
100 **Ans.**

6. Richard has a 9" x 13.5" sheet of cardboard. He wants to create a closed cube with the largest possible volume. He can use all of the cardboard (and won't waste any).
The area of the sheet becomes the surface area of the cube.
$9 \times 13.5 = 121.5$
$121.5 = 6s^2$
$s^2 = \frac{121.5}{5} = 20.25$
$s = \sqrt{20.25} = 4.5$
$s^3 = 4.5^3 = 91.125$ **Ans.**

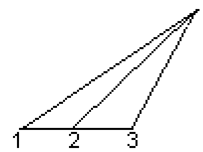
7. The figure below represents a figure with a total of 120 triangles and n points labeled as vertices on the horizontal base. We are asked to figure out what n is.



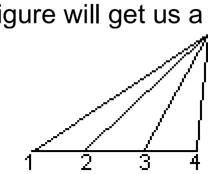

Oh, this is fun…I'm thinking we better find a series in here somewhere…Start with a similar figure where n = 2.




Clearly there is only one triangle. Now look at a similar figure with n = 3.




There are two smaller triangles and the one large triangle. So when n = 3 the total is 3. Hopefully, one more figure will get us a series…n = 4




There are 3 single triangles, two doubles and one large for a total of 6. Yes, yes, yes!!! A series.
n = 2 Total: 1
n = 3 Total: 3
n = 4 Total: 6
$f(n) = f(n-1) + n-1$
$f(5) = 6 + 4 = 10$
$f(6) = 10 + 5 = 15$
$f(7) = 15 + 6 = 21$
$f(8) = 21 + 7 = 28$
$f(9) = 28 + 8 = 36$
$f(10) = 36 + 9 = 45$
$f(11) = 45 + 10 = 55$
$f(12) = 55 + 11 = 66$
$f(13) = 66 + 12 = 78$
$f(14) = 78 + 13 = 91$
$f(15) = 91 + 14 = 105$
$f(16) = 105 + 15 = 120$!!!
n = 16 **Ans.**

8. My three-digit code is 023. Reckha can't choose a code that is the same as mine in two or more of the three digit-positions. This means 02?, 0?3 and ?23 are not allowed. She also can't choose a code that is the same as mine except for switching the positions of two digits. This means that 203, 032, and 320 are also no good. Recha can choose any three-digit code where each digit is in the set {0, 1, 2, …, 9} with the above exceptions. We are asked to determine the number of codes available for Reckha.
There are a total of 1000 possibilities. 02? takes 10 possibilities as does 0?3 and ?23. That's a total of 30 possibilities. But not really. 023 is on each of the three lists so there are 2 extra copies of it. 30 – 2 = 28
28 + 3 = 31
1000 – 31 = 969 **<u>Ans.</u>**

## TEAM ROUND

1. A triangular region is enclosed by the lines with equations
$y = \frac{1}{2}x + 3$
$y = -2x + 6$ and
$y = 1$. Find the area of the triangular region.
The triangular region is formed at the points where the lines meet.
$y = 1$ and $y = \frac{1}{2}x + 3$ meet when
$1 = \frac{1}{2}x + 3$
$2 = x + 6$
$x = -4$
The point is (-4,1).
Now consider the point where
$y = 1$ and $y = -2x + 6$ meet.
$1 = -2x + 6$
$2x = 5$
$x = \frac{5}{2}$
The point is $\left(\frac{5}{2}, 1\right)$.
Finally, consider the point where
$y = \frac{1}{2}x + 3$ and $y = -2x + 6$ meet.
$\frac{1}{2}x + 3 = -2x + 6$
$x + 6 = -4x + 12$
$5x = 6$
$x = \frac{6}{5}$
$y = -2 \times \frac{6}{5} + 6 = \frac{-12}{5} + 6 = \frac{18}{5}$
The point is $\left(\frac{6}{5}, \frac{18}{5}\right)$.
The first two points have y = 1. Thus, we have a straight line. The length of that line is
$\frac{5}{2} - -4 = \frac{5}{2} + 4 = \frac{13}{2}$
The third point is
$\frac{18}{5} - 1 = \frac{13}{5}$ above the line y = 1
and, thus, the height of the triangle is $\frac{13}{5}$.
$A = \frac{1}{2}bh = \frac{1}{2} \times \frac{13}{2} \times \frac{13}{5} = \frac{169}{20}$
$\frac{169}{20} = 8\frac{9}{20}$ = 8.45 **<u>Ans.</u>**

2. Eight apples cost the same as four bananas, and two bananas cost the same as three cucumbers. How many cucumbers can Tyler buy for the price of 16 apples?
Let a = the cost of an apple.
Let b = the cost of a banana.
Let c = the cost of a cucumber.
Then:
$8a = 4b$
$2b = 3c$
$16a = 8b = 12c$
12 **<u>Ans.</u>**

3. A four-digit integer m and the four-digit integer obtained by reversing the order of the digits of m are both divisible by 45. If m is divisible by 7, what is the greatest possible value

of m?
If a number is divisible by 45 it is divisible by 9 and by 5. This means the sum of the digits in m must be divisible by 9 and the number ends in either 5 or 0. The number is also divisible by 7. Since the reverse of m is also divisible by 45, we can't have m end in 0. Because then the reverse of m wouldn't be a four-digit number. It would be, at maximum a three-digit number! So… m ends in 5. Similarly, it has to start with 5 because of reversing the digits.
Let's start by figuring out the largest number less than 6,000 that ends in 5 and is divisible by 45.
5995 is not since 5 + 9 + 9 + 5 = 28 but looking at this tells us that the largest number divisible by 45 must not only end in 5 but the sum of the numbers can't be more than 27 which says that the other 3 numbers will probably sum to 27. Try 5985.
Is that divisible by 7? Yes.

$\frac{5985}{7} = 855$

5985 **Ans.**

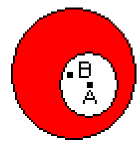
4. A capital "N" can be formed by properly connecting 10 numbers in the chart. Every "N" must be congruent to this one and maintain the same orientation. The sum of the ten numbers of a particular "N" is 525.
Which "N" is this?
Look at the current drawing of "N" and represent 0 by x. Then we can say:
(30 + x) + (20 + x) + (10 + x) + x + (x + 11) + (x + 22) + (x + 33) + (x + 23) + (x + 13) + (x + 3) = 525
10x + 165 = 525
10x = 360
x = 36 And that's the least of the 10 numbers! 36 **Ans.**

5. $\frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdot \ldots \cdot \frac{n}{n+1} = \frac{1}{50}$
Everything is going to cancel out on the left hand side of the equation except the "2" and the "n + 1".

$\frac{2}{n+1} = \frac{1}{50}$

n + 1 = 100
n = 99
99 + 100 = 199 **Ans.**

6. Circle A is in the interior of circle B. The diameter of circle B is 16 cm. What is the diameter of circle A for which the ratio of the shaded area to the area of circle A is 3:1?




The area of circle B is
$\pi \times 8^2 = 64\pi$
Let x be the area of circle A.
Then the area of the area shaded in red is 64π – x
64π – x = 3x
64π = 4x
x = 16π
Let r = the radius of circle A.
$16\pi = \pi r^2$
r = 4
2r = 8 **Ans.**

7. In the game of Frood, dropping n froods gives a score of the sum of the first n positive integers. Eating n froods earns 10n points. We must determine the least number of froods for which dropping them will earn more points than eating them. (I don't know about you but unless froods are like pizza or candy I'm not really sure I'd want to play but if they were, I know I wouldn't care about winning and would just eat those yummy froods…Anyway…)
Until we start adding more than 10 points at a time in dropping froods, we can't hope to catch up. How far behind will we be if we drop 10 froods instead of eating them?
1 + 2 + … + 10 = 11 × 5 = 55
10 × 10 = 100
Hmmm… 45 to go… From here on in we will start to catch up. 1 point the next time, 1 + 2 = 3 points after that. To catch up by 45, we need to go 9 times (since 1 + 2 + + 9 = 45).

Dropping 19 froods gives us

$1 + 2 + \ldots + 18 + 19 = 20 \times 9\frac{1}{2} = 190$

$19 \times 10 = 190$

The 20th time dropping will win.

20 **Ans.**

8. Kendra can move right, left, down or up on the game board. (I don't think I'm going to draw this one…) We are asked to find the probability that the sum of the numbers in the spaces on which he will land will be exactly 30 after her third complete turn.
There are 4 different ways you can go at each turn or
$4 \times 4 \times 4 = 64$ combinations. How many combinations sum to 30?
We can do it either with 10 + 10 + 10, or 10 + 5 + 15 because getting to a 20 takes at least 4 turns. To get 10 + 5 + 15 we can go: RRU; RRD; LLU or LLD. That's 4 combinations. (R = right, U = up, D = down, L = left)
To get 10 + 10 + 10, look at one scenario first (first move up): URD; URL; ULD; ULR. Four similar patterns emerge with a first move of R, D or L. This gives us a total of 16 combinations for 10 + 10 + 10.
16 + 4 = 20 total combinations.
$\frac{20}{64} = \frac{5}{16}$ **Ans.**

9. Emma places her square unit tiles into different shaped rectangular figures. She can form exactly 10 different rectangular figures that use all of her tiles. What's the least number of tiles that Emma could have? The area of a rectangle is just the number of tiles that Emma has and the area is computed as length times width which says that the number of tiles that Emma has has 20 factors (including 1 and the number itself). Don't they think they're clever? This is a factoring problem! Since we're looking for the minimum number, let's start with 1, 2, and 3 as factors and try to add the smallest numbers possible.
Then we have a total of 4 factors, i.e., 1, 2, 3, 6
If we want 4 as a factor, we can add another 2, i.e., 1, 2, 2, 3
Then we have a total of 6 factors.
1, 2, 3, 4, 6, 12
Now add 5 (1, 2, 2, 3, 5) for a total of 12 factors.
1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60
We're getting there…
Try adding 7 (1, 2, 2, 3, 5, 7) for a total of 22 factors. Oops, can't use 7.
1, 2, 3, 4, 5, 6, 7, 10, 12, 15, 20, 21, 28, 35, 42, 60, 70, 84, 105, 140, 210, 420
For 8, we need only to add another 2. (1, 2, 2, 2, 3, 5) for 16 factors.
1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120
Better. Try 9. For that we need another 3. (1, 2, 2, 2, 3, 3, 5)
1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, 360 for a total of 24 factors. Too much. Remove the 3.
Next is 10, but we already have it.
At this point it seems safe to stay away from primes; we'll get too many factors. Forget 11. 12 is next but we already have that. Forget 13. 14 is next but requires a 7 and we already know that that is no good. 15 we have but not 16. Add another 2 (1, 2, 2, 2, 2, 3, 5)
1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 18, 20, 24, 30, 40, 48, 60, 80, 120, 240
and there we have 20 factors!!!
240 **Ans.**

10. In a math class, 12 out of 15 girls are freshmen and 11 out of 15 boys are freshmen. What is the probability that in a randomly selected group of five students from the class, there will be two freshmen girls and three freshmen boys?
$\frac{12}{30} \times \frac{11}{29} \times \frac{11}{28} \times \frac{10}{27} \times \frac{9}{26} = \frac{130680}{17100720}$
This is the probability of choosing GGBBB, but we weren't told that the order mattered. So there are

$\frac{5!}{3!2!} = \frac{5 \times 4}{2 \times 1} = 10$ different orders that could be used in addition to this one.

$\frac{130680}{17100720} \times 10 = \frac{1306800}{17100720} \approx$ .076417 ≈ .076 **Ans.**