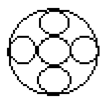
2007
MATHCOUNTS STATE COMPETITION

SPRINT ROUND

1. In the figure, the largest circle has a radius of six meters. Five congruent smaller circles are placed as shown lined up in east-to-west and north-to-south orientations. We are asked to find the radius of one of the smaller circles.

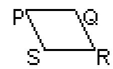


The radius of the large circle is 6. Thus, the diameter of the large circle is 6 × 2 = 12. The diameter runs through the 3 congruent smaller circles lined up in the east-to-west orientation. This means that the diameter of each circle is $\frac{12}{3}$ = 4. The radius is just half that, or 2. **Ans.**

2. Two identical CDs cost a total of $28. This means that one CD costs $\frac{1}{2}$ × 28 = 14. And 5 CDs must cost 5 × 14 = 70 **Ans.**

3. Alexandra played 15 games of Skee Ball for $11.25. She earned either 2 tickets or 4 tickets for playing each game and after playing the 15 games had a total of 40 tickets. We are asked to find out how many games she played where she earned 4 tickets.
Let x = the number of games she played where she got 4 tickets.
Let y = the number of games she played where she got 2 tickets.
x + y = 15 Eq. 1
4x + 2y = 40 Eq. 2
4x + 4y = 60 Eq. 3 = Eq. 1 * 4
2y = 20 Eq. 3 – Eq. 2
y = 10
x = 15 – y = 15 – 10 = 5
5 **Ans.**

4. In parallelogram PQRS, the measure of angle P is five times the measure of angle Q. A parallelogram has two sets of opposite parallel lines. The angles opposite each other are the same.



The sum of angle P and angle Q must be 180º. Let x = angle Q. Then, 5x = angle P.
5x + x = 180
6x = 180
x = 30
5x = 150
Angle R = Angle P = 150 **Ans.**

5. Two right triangles have equal areas. The first triangle has a height of 5 cm and a corresponding base of 8 cm. That means that its area is $\frac{1}{2}$ × 5 × 8 = 20
The second triangle has a side of length 20. We know that its area is also 20 but we don't know its height. Let h = the height of the second triangle.
20 = $\frac{1}{2}$ × 20 × h
20 = 10h
h = 2 **Ans.**

6. In a class, 12 students are girls out of a total of 30 students. Therefore, 30 – 12 = 18 students are boys. 6 more boys join the class for a new total of 18 + 6 = 24 boys. The fraction of the class that is now boys is:
$\frac{24}{30+6} = \frac{24}{36} = \frac{2}{3}$ **Ans.**

7. The distance from the earth to the moon is approximately 2.4 × 10^5 miles. A satellite crosses the shortest path between the earth and the moon at a distance of 240 miles from the earth. We are asked to find the fraction of the distance from the earth to the moon that the satellite's distance from the earth is at this point.

$$\frac{240}{2.4 \times 10^5} = \frac{2.4 \times 10^2}{2.4 \times 10^5} =$$
$$\frac{1}{10^3} = \frac{1}{1000}$$ **Ans.**

8. How many perfect squares less than 1000 have a ones digit of 2, 3 or 4? Which digits when squared have a ones digit of 2? Not anything odd; that's for sure. In the even values we have 2 × 2 = 4, 4 × 4 = 16, 6 × 6 = 36 and 8 × 8 = 64. Don't even bother with 0! Doesn't look like any perfect squares can end in 2.
How about 3? We'll need to look at odd numbers only and obviously, don't even look at 1! 3 × 3 = 9, 5 × 5 = 25, 7 × 7 = 49 and 9 × 9 = 81. So there aren't any that end in 3 either.
How about 4? When we looked at even squares original we did see that squares coming from values with the ones digit being either 2 or 8 will result in a square that has 4 for its ones digit.
What's the largest value that when squared is still <= 1000?
$30^2 = 900$; that's okay
$31^2 = 961$; that's okay
32 will be too large. Thus, any positive integer less than 32 that ends in 2 or 8 will satisfy the requirements. Let's list them:
2, 8, 12, 18, 22, 28
There are 6 of them. 6 **Ans.**

9. Over time, a patient needs to get allergy shots totaling 60 mL. The first 20 shots are each 0.5 mL for a total of 20 × 0.5 = 10 mL. The next 20 shots are each 1 mL for a total of 20 mL. That's 10 + 20 = 30 mL so far. Finally, all the others are 2 mL after that.
60 – 30 = 30
$\frac{30}{2} = 15$
Thus, the total number of shots needed are 20 + 20 + 15 = 55 **Ans.**

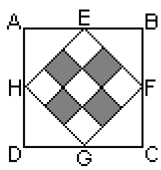
10. Louisa runs at an average speed of 5 mph along a circular park path. Calvin runs in the opposite direction at an average speed of 6 mph. We are told that it takes Calvin 30 minutes less than Louisa to run the full path once and we are asked to find out how many miles Louisa ran when she completed one circular path.
Let x = the number of minutes it takes Calvin to go all the way around. Then it takes Louisa x + 30 minutes to go all the way around.
Since Calvin averages 6 mph, each minute he goes $\frac{6}{60} = \frac{1}{10}$ mile
Louisa averages 5 mph, so each minute she goes $\frac{5}{60} = \frac{1}{12}$ mile
$\frac{1}{10}x = \frac{1}{12}(x + 30)$
$12x = 10(x + 30)$
$12x = 10x + 300$
$2x = 300$
$x = 150$
$\frac{1}{10} \times 150 = 15$ **Ans.**

11. The cost of renting a bus for a field trip was split evenly among 20 students. Then 10 more students joined the trip. The cost of renting the bus was then evenly redistributed among all of the students. The cost for the original students decreased by $1.50 and we are asked to find the total cost of renting the bus. Let x = the cost of renting the bus. Then the 20 students were each going to pay $\frac{1}{20}$ of the cost of the bus. But after the 10 other students came each share came to $\frac{1}{30}$ of the total cost of the bus.
$$\frac{1}{20}x - \frac{1}{30}x = \frac{3}{60}x - \frac{2}{60}x = \frac{1}{60}x$$
Thus the difference that the original 20 students got back, or $1.50 is

actually $\frac{1}{60}$ of the cost of the bus.
60 × \$1.50 = \$90 **Ans.**

12. The date is March 1, 2007. In how many whole years will the number representing the year be the smallest perfect square greater than 2007?
Let's try and estimate this easily first.
$30^2 = 900$; $40^2 = 1600$; $50^2 = 2500$
Thus, the integer we must square is somewhere between 40 and 50.
$45^2 = 2025$ and that's exactly it!
44 is going to be too small.
2025 – 2007 = 18 **Ans.**

13. Points E, F, G, and H are the midpoints of the sides of square ABCD, and square EFGH is divided into nine congruent unit squares, as shown.



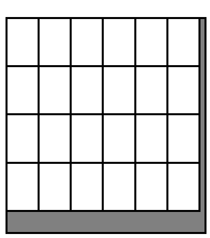
We are asked to determine the percentage of the total area of square ABCD that is represented by the shaded area. If the side of square ABCD is x, then the side of square EFGH is $\frac{x}{2}\sqrt{2}$ and its area is $\frac{x^2}{2}$ or $\frac{1}{2}$ the area of square ABCD. Since square EFGH is broken into 9 equal squares, each of those is $\frac{1}{9}$ the area of square EFGH or $\frac{1}{18}$ the area of square ABCD.
Four of the inner squares are shaded.
$\frac{4}{18} = \frac{2}{9}$ = 22.222…%
22 **Ans.**

14. 20 people are seated around a circular table. Each person is either a knight, who always tells the truth, or a knave, who always lies. Each person says "The person on my right is a knave. How many knaves are seated around the table?
Hmm… let's take a knight. If he says the person seated next to him is a knave, then it must be knave. So we now move to the knave who also says the same thing so we know the next person sitting next to him is a knight. As you can see the knights and knaves are all sitting adjacent to each other – knight, knave, knight, knave etc. Doesn't matter if you start with a knave, you'll still get the same answer.
$\frac{1}{2}$ × 20 = 10 **Ans.**

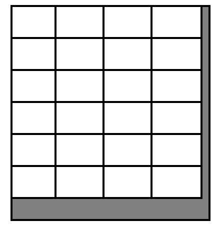
15. There is one scheduled appointment every five minutes starting at noon, at the clinic. Each patient arrives at their scheduled time, spends 10 minutes in the lobby doing paperwork, five minutes out of the lobby getting shots, and 20 more minutes back in the lobby to check for an adverse reaction. We must find the number of patients in the lobby at 12:34 p.m.
The person who arrives at noon gets their shot at 12:10, is back in the lobby at 12:15 and leaves at 12:35. That person is still there at 12:34.
Clearly, the person who comes in at 12:05 returns to the lobby at 12:20. That's 2.
The person who comes in at 12:10 returns to the lobby at 12:25. That's 3.
The person who comes in at 12:15 returns to the lobby at 12:30. That's 4.
The person who comes in at 12:20 is getting their shot from 12:30 to 12:35. This person doesn't count.
The person who comes in at 12:25 is still in the lobby at 12:34 as is the person who arrives at 12:30. That's 2 more.
4 + 2 = 6 **Ans.**

16. The ages of the 25 sixth graders are either 10 or 11. The median age in the data set is 0.36 years greater than the mean.
Let x = the number of 10 year olds.
Let y = the number of 11 year olds.
The mean is $\frac{10x + 11y}{25}$
x + y = 25
The median can be either 10, or 11. It cannot be 10.5 since there are an odd number of values and we will choose the 13th value as the median. Since the median is greater than the mean and the mean can't be less than 10, we can remove 10 as the median. This leaves us with only 1 possibility: 11.
$11 - 0.36 = \frac{10x + 11y}{25}$
$10.64 = \frac{10x + 11y}{25}$
y = 25 – x
$10.64 = \frac{10x + 11(25 - x)}{25}$
10x + 275 – 11x = (10.64 × 25)
-x + 275 = 266
x = 9 **Ans.**

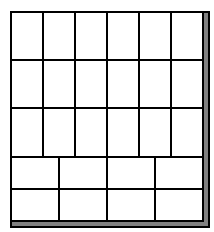
17. We are asked to find the greatest number of whole 4-inch by 6-inch notecards that can be cut from a 25-inch by 27-inch sheet of poster board. If we place the 4-inch side of the notecard on the 25-inch side of the poster board, we'd get 6 across with wastage of 1 inch, and 4 down with a wastage of 3 inches. That's a total of 24 cards.



And if we turn them 90 degrees?



Well, that doesn't get us anything better. Normally, these type of problems fill up the entire space. This one doesn't, and since it's a state competition, I'm a little suspicious that there might be a better way to place things. I know that it's impossible to use up all the space. After all, we're adding even values to even values and we'll never get an odd value that way. But it would be nice to limit the loss to 1 inch on a side. Such a solution is the following:



What I did is to look for combinations of 6 and 4 that add up to 26. And certainly (6 × 3) + (4 × 2) = 26. Therefore, we have 18 + 8 = 26 cards. 26 **Ans.**

18. Brand X soda gives you 20% more soda than Brand Y for a price that is 10% less than Brand Y's price. We are asked to find the ratio of the unit price of Brand X soda to the unit price of Brand Y soda.
Let Brand Y give you y ounces of soda. Then Brand X gives you 1.2y ounces of soda. Let z be the price of Brand Y's soda. Then .9z is the price of Brand X's soda. The price per ounce for Brand Y is $\frac{z}{y}$. The price per ounce for Brand X is
$\frac{.9z}{1.2y} = \frac{3z}{4y}$

The ratio of the unit price of Brand X to Brand Y is

$$\frac{\frac{3}{4}\frac{z}{y}}{\frac{z}{y}} = \frac{3}{4}$$ **Ans.**

19. We are given a 2 by 2 grid of digits where the digits in each row, from left to right, form a two-digit perfect square. They also do the same in each column from top to bottom. So how many of these grids are there?
First, list two-digit perfect squares.
16, 25, 36, 49, 64, 81
Clearly, whatever the first row's digits are, the second one (the ones digit) must also be a tens digit in another perfect square. That let's out 25 and 49 since no two-digit perfect square starts with 5 or 9. This leaves 16, 36, 64, and 81 to deal with. And the solutions are:

1	6
6	4

3	6
6	4

6	4
4	9

8	1
1	6

There are 4 grids. 4 **Ans.**

20. A sheet of notebook paper weighs 0.005 ounces per square inch. A box contains 24 notebooks each of which as 100 sheets of 10-inch by 6-inch paper. The back and front covers together weigh one-half ounce. We are asked to find out how much the total weight of the notebooks in the box is.
Start by finding out how much a single piece of paper weighs. There are 10 × 6 = 60 square inches of paper. At 0.005 ounces per square inch the paper weighs 60 × 0.005 = 0.3 ounces.
There are 100 sheets of paper in a notebook. The 100 sheets weight 0.3 × 100 = 30 ounces.
The cover weighs 0.5 ounces so a single notebook weighs 30 + 0.5 = 30.5 ounces.
There are 24 notebooks.
24 × 30.5 = 732 **Ans.**

21. The product of two positive fractions is $\frac{1}{9}$. The larger fraction divided by the smaller fraction is 4.
Let x be the larger fraction.
Let y be the smaller fraction. Then:

$xy = \frac{1}{9}$

$\frac{x}{y} = 4$

$x = 4y$

$4y \times y = \frac{1}{9}$

$4y^2 = \frac{1}{9}$

$y^2 = \frac{1}{36}$

$y = \frac{1}{6}$

$x = 4y = \frac{4}{6} = \frac{2}{3}$

$x + y = \frac{2}{3} + \frac{1}{6} = \frac{5}{6}$ **Ans.**

22. We are asked to determine how many outcomes of rolling 5 6-sided dice result in the sums of the values on the dice being exactly 27. Let's list the number of ways to make 27.
6 + 6 + 6 + 6 + 3 (in no particular order)
6 + 6 + 6 + 5 + 4
That's as far as we can go with 3 6's and not repeat ourselves.
6 + 6 + 5 + 5 + 5
And we can't go any further with 2 6's.
And we're done, too. The number of ways to roll 4 6's and one 3 is:

$$\left(\frac{5!}{4!1!}\right) = 5$$

The number of ways to roll 3 6's a 5 and a 4 is:

$$\left(\frac{5!}{3!1!1!}\right) = 20$$

The number of ways to roll 2 6's and 3 5's is:

$\left(\frac{5!}{2!3!}\right) = 10$

5 + 20 + 10 = 35 **Ans.**

23. The probability of selecting a red marble from a box containing some green marbles and 4 red marbles is x%. If the number of green marbles is doubled, the probability of selecting one of the four red marbles from the box is (x – 15)%. We are asked to determine how many green marbles were in the box before the number of green marbles was doubled.

Let y = the original number of green marbles. Then:

$x\% = \frac{x}{100} = \frac{4}{4+y}$

$(x-15)\% = \frac{x}{100} - \frac{15}{100} = \frac{4}{4+2y}$

$\frac{4}{4+y} - \frac{15}{100} = \frac{4}{4+2y}$

$\frac{3}{20} = \frac{4}{4+y} - \frac{4}{4+2y}$

$\frac{3}{20} = \frac{4(4+2y) - 4(4+y)}{(4+y)(4+2y)}$

$\frac{3}{20} = \frac{16+8y-16-4y}{16+8y+4y+2y^2}$

$\frac{3}{20} = \frac{4y}{2y^2+12y+16}$

$80y = 6y^2 + 36y + 48$

$6y^2 - 44y + 48 = 0$

$3y^2 - 22y + 24 = 0$

$(3y - 4)(y - 6) = 0$

3y = 4 won't work since we can't get an integer out of it

But y = 6 will **Ans.**

24. Julie iced 4 cupcakes with red frosting, 2 cupcakes with orange frosting, 2 with yellow, 2 with green, 3 with blue and 3 with violet frosting. She will take 10 cupcakes to the party and if she takes one of a particular color, she must take all of them. The number of different combinations of cupcakes comes down to how many ways we can add the numbers of same-colored cupcakes to make 10.

We have 4, 2, 2, 2, 3, 3, respectively.

1--4 red 2 orange 2 yellow 2 green

2--4 red 3 blue 3 violet

Any other ways to use 4? No.

Can we use all the 2's? No, since that is 6 and we only have 2 3's left.

How about 2 2's? Yes.

3--2 orange 2 yellow 3 blue 3 violet

4--2 orange 2 green 3 blue 3 violet

5--2 yellow 2 green 3 blue 3 violet

How about 1 2? No, since we'd only have 2 3's left.

We can't use a single 3 either because all the other numbers are even.

So that makes 5 ways. 5 **Ans.**

25. Monica reduced $\frac{154}{253}$ to a common fraction by canceling the middle 5s to get $\frac{14}{23}$. (I think I'd like to meet her math teacher!) The reduction is actually correct though. Kia's example, $\frac{242}{x}$, also works by removing the middle digits. (x is a 3 digit integer). We are asked to find the greatest possible value for x.

Clearly, 242 and x must have some common divisors so let's start by factoring 242.

242 = 121 × 2 = 11 × 11 × 2

x must be a factor of either 11 and/or 2.

If x = abc where a is the hundreds digit, b is the tens digit and c is the ones digit, then ac should not be a multiple of either 11 or 2 or we would have to reduce this further. The middle digit also has to match so b = 4. So what values could it be?

946, 847, 748, 649, 440, 341, 242 and 143. Several of these are even so we can get rid of them leaving: 847, 649, 341 and 143. Since we're looking for the largest, let's start with 847.

$$\frac{242}{847} = \frac{22 \times 11}{77 \times 11} = \frac{22}{77}$$

This isn't the same as $\frac{22}{87}$ so 847 doesn't work. Next is 649.

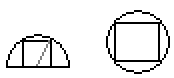
$$\frac{242}{649} = \frac{22 \times 11}{59 \times 11} = \frac{22}{59}$$

This isn't the same as $\frac{22}{69}$ so 649 is out. Next is 341.

$$\frac{242}{341} = \frac{22 \times 11}{31 \times 11} = \frac{22}{31}$$

We have a winner! x = 341 **Ans.**

26. The area of the semicircle in the first figure is half the area of the circle in the second figure.



We are asked to determine what fraction the area of the square inscribed in the semicircle is of the area of the square inscribed in the circle.
The fact that the area of the semicircle is half the area of the circle says that the radius is the same for both the semicircle and the circle. Let y be that radius.
Then 2y is the diagonal of the larger square.
Let s be the side of the larger square which makes s^2 the area.
$s^2 + s^2 = (2y)^2$
$2s^2 = 4y^2$
$s^2 = 2y^2$
For the square inscribed in the semicircle, let x be the side of the square. If you draw a line from the center of the circle to the point at which the square meets the circle (the grey line) you have a radius. From there, the vertical line going back to the semicircle diameter is x, the size of the square. And the final line which goes back to the center of the circle is just one half of the side of the square.

$$x^2 + \left(\frac{x}{2}\right)^2 = y^2$$

$$x^2 + \frac{1}{4}x^2 = \frac{5}{4}x^2 = y^2$$

$$x^2 = \frac{4}{5}y^2$$

The ratio of the smaller to the larger square is:

$$\frac{\frac{4}{5}y^2}{2y^2} = \frac{\frac{4}{5}}{2} = \frac{2}{5}$$ **Ans.**

27. A collection of 20 coins is made up of only nickels, dimes and quarters. It has a total value of \$3.35. If the dimes were nickels, the nickels were quarters and the quarters were dimes, the collection of coins would have a total value of \$2.75. (Just who makes up these questions, anyway?) We are asked to find how many quarters are in the original collection.
Let n, d, and q be the original number of nickels dimes and quarters, respectively. Then:
5n + 10d + 25q = 335
25n + 5d + 10q = 275
n + d + q = 20
Multiply the previous equation by 5.
5n + 5d + 5q = 100
Subtract it from the first equation.
5d + 20q = 235
d + 4q = 47
d = 47 – 4q
Substitute into the second equation:
25n + 5(47 -4q) + 10q = 275
25n + 235 – 20q + 10q = 275
25n – 10q = 40
5n – 2q = 8
5n = 2q + 10

$$n = \frac{2q + 8}{5}$$

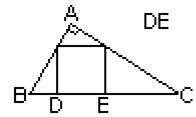
$$\frac{2q + 8}{5} + 47 - 4q + q = 20$$

2q + 8 + 235 – 20q + 5q = 100
-13q + 243 = 100
13q = 143
q = 11 **Ans.**

28. Steve was born in 1950. He had a grandmother who was born in a year that is the product of two prime numbers, one of which is one less than twice the other. We need to find the year in which she reached her 90th birthday. We can assume that Steve's grandmother was born somewhere in the late 1800's or early 1900's.
Let x be the first prime number.
Then the second prime number is 2x – 1.
Hmmm… Why don't we try some estimating? If x were 41,
41 × (82 – 1) = 41 × 81 = 3321
Way too big.
How about 31?
31 × (62 – 1) = 31 × 61 = 1891
Now that looks pretty reasonable.
What's the next largest prime?
37 so 37 × (74 – 1) = 37 × 73 = 2701.
Nah. So Steve's grandmother was born in 1891. 1891 + 90 = 1981
Ans.

29. The digits 2, 3, 4, 7 and 8 will be put in random order to make a positive five-digit integer. We are asked to find the probability that the resulting integer will be divisible by 11.
The way to tell if a number is divisible by 11 is the following:
If the sum of every other digit, starting with the first is equal to the sum of every other digit, starting with the second, then the number is evenly divisible by 11.
Our given numbers are 2, 3, 4, 7 and 8. Which 3 numbers summed are equivalent to the sum of the other 2?
By inspection we can see that 2 + 3 + 7 = 12 and 8 + 4 equal 12.
There are 3! = 6 ways to use the 2, 3, and 7 and 2! = 2 ways to use the 8 and 4.
6 × 2 = 12.
There are 5! = 120 different possibilities
$\frac{12}{120} = \frac{1}{10}$ **Ans.**

30. Right triangle ABC has one leg of length 6 cm, one of length 8 cm and a right angle at A. The square inscribed within the triangle has one side on the hypotenuse of ABC and a vertex on each of the two legs of ABC. So what is the length of one side of the square?



We know that triangle ABC is a 6-8-10 right triangle and that the smaller triangle containing angle A is similar to triangle ABC. The area of triangle ABC is $\frac{1}{2} \times 6 \times 8 = 24$
Let s = the side of the square.
Let x = BD
Let y = EC
Then x + s + y = 10.
The area of the right triangle with sides s and x is $\frac{1}{2}$xs.
The area of the right triangle with sides s and y is $\frac{1}{2}$ys.
If you drop a perpendicular from point A to BC to create the height, we also know that h = 4.8 and we can compute the area of the triangle with point A as $\frac{1}{2}(4.8 - s)s$
The area of the square is s^2. The sum of the 3 smaller triangles and the square is 24.
$\frac{1}{2}xs + \frac{1}{2}ys + \frac{1}{2}(4.8 - s)s + s^2 = 24$
$\frac{1}{2}xs + \frac{1}{2}ys + 2.4s - \frac{1}{2}s^2 + s^2 = 24$
$\frac{1}{2}s(x + y) + 2.4s - \frac{1}{2}s^2 + s^2 = 24$
$\frac{1}{2}s(10 - s) + 2.4s - \frac{1}{2}s^2 + s^2 = 24$
$5s - \frac{s^2}{2} + 2.4s - \frac{s^2}{2} + s^2 = 24$
5s + 2.4s = 24
(Boy am I glad the s^2 terms went away!!!)
7.4s = 24

74s = 240

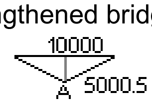
s = $\frac{240}{74} = \frac{120}{37}$ **Ans.**

Whew!

TARGET ROUND

1. A recipe for crispy rice treats results in a mixture that fills a 9-inch by 13-inch pan to a depth of one inch. If we make 1.5 times the original recipe and pour it into a pan that is 10 inches by 15 inches, then to what depth is the pan filled?
A single recipe filled a 9-inch by 13-inch pan to a depth of one inch.
That's 9 × 13 = 117 cu. inches.
If we make 1.5 times the recipe, that's 117 × 1.5 = 175.5 cu. inches.
But we're going to pour this into a 10 by 15 inch pan. This pan, obviously holds 150 cu. inches for the first inch.
$\frac{175.5}{150}$ = 1.17 **Ans.**

2. Suppose a bridge 10,000 feet in length expands by one foot on a warm day. Given that the side supports do not move, the middle of the bridge sags down, making two straight congruent line segments as shown. We are asked to determine how many feet below the original bridge's height is the middle of the lengthened bridge.



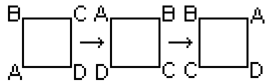
This forms two right triangles. The length of the original bridge is bisected, extending a perpendicular from point A to the original bridge span divides 10000 in 2. So the one side of a triangle is 5000. Since the original size was stretched by one foot to 10001 feet, each hypotenuse of both triangles is half of that, or 5000.5. Let h be the perpendicular. Then:
$5000^2 + h^2 = (5000.5)^2$
$25000000 + h^2 = 25005000.25$
$h^2 = 25005000.25 – 25000000 = 5000.25$
h = 70.71244586… ≈ 71 **Ans.**
(That's one really bad design for a bridge, isn't it?)

3. One cube has a volume of 343 cubic centimeters. The edge length of a second cube is 3 times the edge length of the first cube. We are asked to find the positive difference between the volume of the second cube and the volume of the first cube.
$343 = 7^3$
Therefore, the edge length of the first cube is 7 and the edge length of the second cube is 7 × 3 = 21.
$21^3 = 9261$
9261 – 343 = 8918 **Ans.**

4. We are asked to find in how many ways 36 can be written as the product a × b × c × d, where a, b, c and d are positive integers such that a ≤ b ≤ c ≤ d.
First factor 36:
36 = 2 × 2 × 3 × 3
The factors of 36 are:
1, 2, 3, 4, 6, 9, 12, 18 and 36
Suppose the highest value we use is 36.
1 × 1 × 1 × 36
Can't do anything more with 36.
Now 18.
1 × 1 × 2 × 18
Now 12
1 × 1 × 3 × 12
Now 9:
1 × 1 × 4 × 9
1 × 2 × 2 × 9
Now 6:
1 × 1 × 6 × 6
1 × 2 × 3 × 6
Now 4:
1 × 3 × 3 × 4
Now 3:
2 × 2 × 3 × 3
We're done. Let's count them up:
9 **Ans.**

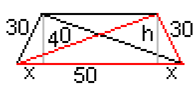
5. The first square is in position ABCD. After rotating the square 90 degrees clockwise about its center point, the second square is in position DABC. Reflect DABC over its vertical line of symmetry, and this will result in the square CBAD. If we repeat this

pattern of alternately rotating 90 degrees clockwise and reflecting over the vertical line of symmetry, then in what position will the 2007th square be?



Let's hope we can find a repetition! The fourth move is a 90 degree rotation which leaves us with DCBA. The fifth is a reflection which leaves us with ABCD!!!!! It looks to me that every 4 moves will bring us back to the beginning. Whew. So the values 1, 5, 9, etc. will be the beginning. What is 2007 mod 4?
Well, it's 3. That means it's the third triangle in the pattern or CBAD. **Ans.**

6. An isosceles trapezoid has legs of length 30 cm each. It also has diagonals of length 40 cm each and its longer base is 50 cm.



Drop two perpendicular from either end of the shorter base (either of the grey lines) and call them both h. The two perpendiculars intersect the longer base creating 3 line segments of length x, 50 – 2x, and x. Note also that what we've created 2 right triangles of size 30, 40, 50 (one of which is shown in red). The area of one of these triangles is:

$\frac{1}{2} \times 30 \times 40 = 600$

But this is the same as

$\frac{1}{2} \times 50 \times h$

$25h = 600$
$h = 24$
Now that we know what h is, let's figure out what x is.
$h^2 + x^2 = 30^2$
$24^2 + x^2 = 30^2$
$576 + x^2 = 900$
$x^2 = 900 – 576 = 324$
$x = 18$
Thus the smaller base is $50 – 2x = 50 – 36 = 14$

$A = \frac{1}{2}(b_1 + b_2)h = \frac{1}{2}(50 + 14)24 =$
$12 \times 64 = 768$ **Ans.**

7. When the expression $(2^1)(2^2)(2^3)\ldots(2^{99})(2^{100})$ is written as an integer, what is the product of the tens digit and the ones digit? First of all, we know that powers of 2 always repeat themselves in the one digits in the order ,2, 4, 8, 6, but what about the tens digit? Let's start looking at powers of 2.
2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192, 16384, 32768, 65536, 131072, 262144, 524288, 1048576, 2097152, 4194304, 8388608, 16777216, 33554432, 67108864
No wonder they let you have a calculator. There's the repeat finally! If we look at the first, 5th, 9th, 13th values in the series, looking at only the tens and one digits, we see 02, 32, 12, 92, 72, 52, 32 and the series will now repeat. This says that the series repeats every every 20 terms starting with the 5th term. So, exactly what power of 2 do we have?
$(2^1)(2^2)(2^3)\ldots(2^{99})(2^{100}) =$
$2^{(1 + 2 + 3 + ,,, 99 + 100)} = 2^{(50 \times 101)} =$
2^{5050}
We have 5050 terms. The 5th term starts the series which repeats every 20 terms.
$5050 – 5 = 5045$

$\frac{5045}{20} = 252 \text{ R } 5$

This must be the 5th term when we start counting from the 5th term. Thus the 5th term is the 10th term in the series so the tens and ones digits are 2 and 4, respectively.
$2 \times 4 = 8$ **Ans.**
Whew!!!!

8. A teenager is someone who is in the range of 13 to 19 years old. The product of the ages of a particular group of teenagers is 705,600. We must find the mean of their ages. First thing to do is to factor 705,600

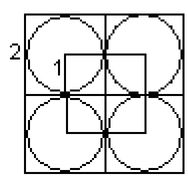
into its prime factors.
705600 = 5 × 2 × 5 × 2 × 7056 =
$2^4 \times 5^2 \times 1764$ =
$2^6 \times 5^2 \times 441$ =
$2^6 \times 3^2 \times 5^2 \times 49$ =
$2^6 \times 3^2 \times 5^2 \times 7^2$ =
Now let's make as many numbers in the range of 13 to 19 as we can.
$14 \times 14 \times 2^4 \times 3^2 \times 5^2$ =
14 × 14 × 15 × 15 × 16
The mean is:

$$\frac{14+14+15+15+16}{5} = \frac{74}{5} =$$

14.8 **Ans.**

TEAM ROUND

1. Four circular chips are each centered on one of four adjacent squares on a checkerboard. The centers of the chips are the four vertices of a square. The checkerboard's squares each measure two inches on a side and we are asked to find the area of the inner square. The figure looks like this:



Given that the side of each square is 2, then the radius of each inscribed circle is 1. The side of the square is 2 and the area is
2 × 2 = 4 **Ans.**

2. A sculpture of a clothespin is 20 feet high. A normal clothespin of this shape is 5 inches high. So how many feet tall would a sculpture of a 5-foot, 7-inch woman be? First change 20 feet into inches.
20 × 12 = 240
5 feet 7 inches = 67
Let x be the size of the sculpture of the woman.

$$\frac{5}{240} = \frac{67}{x}$$

5x = 67 × 240
x = 67 × 48 = 3216
Remember, we have to change this back into feet.

$$\frac{3216}{12} = 268$$ **Ans.**

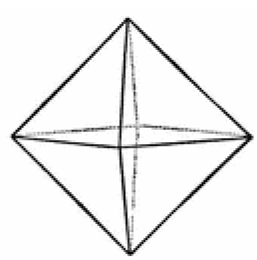
3. All ID codes are three-digit positive integers. The product of the digits in Agueleo's ID code is 216, the sum of the digits is 19 and the integer ID code is as large as possible. So what is it?
Let x, y and z be the three integers that make up the code.
x × y × z = 216
x + y + z = 19
Let's factor 216.
216 = 2 × 2 × 54 =
2 × 2 × 2 × 3 × 3 × 3
By inspection,
4 + 6 + 9 = 19 and
4 × 6 × 9 = 216
But we must find the largest code, so let's just flip this around.
964 **Ans.**

4. A subtraction square is a 3 by 3 grid of integers where, in each row from left to right, and in each column from top to bottom, the first integer minus the second integer equals the third integer. Let's fill it out.

k	x	k – x
y	z	y – z
k – y	x – z	k – y – x + z

Let's add them up.
k + x + k – x + y + z + y – z + k – y + x – z + k – y – x + z = 4k **Ans.**

5. Two right pyramids with congruent square bases and equilateral triangular faces and be joined at their bases to form an octahedron with 8 congruent, equilateral triangular faces.



The total surface area of this octahedron with edge length 2 units

is x square units and the volume is y cubic units. So what is x + y?
If A is the area of one of the equilateral triangular faces then 8A is the surface area.
The area of an equilateral triangle is $\frac{\sqrt{3}}{4}s^2$ where s is the length of the side of the triangle and we know this is 2.

$8 \times \frac{\sqrt{3}}{4} \times 2^2 = 8\sqrt{3} = x$

So this is our surface area.

$y = V = 2 \times \frac{1}{3} \times B \times h$

The base of the right pyramid is a square with sides of 2. Thus the base is 4.

$y = \frac{2}{3} \times 4 \times h = \frac{8}{3}h$

If you drop a perpendicular from the apex of the octahedron to the base at which both pyramids are joined you have a right triangle with sides, h and $\frac{1}{2}$ of the diagonal of the square.
The hypotenuse is just 2.
The diagonal is $2\sqrt{2}$.

$h^2 + \sqrt{2}^2 = 2^2$

$h^2 + 2 = 4$

$h^2 = 2$

$h = \sqrt{2}$

$y = \frac{8}{3}h = \frac{8}{3}\sqrt{2}$

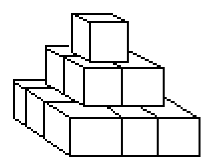
$x + y = 8\sqrt{3} + \frac{8}{3}\sqrt{2} =$

13.85640646… + 3.771236166 ≈ 17.627… ≈ 17.6 **Ans.**

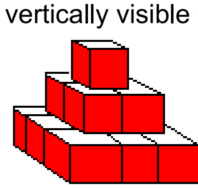
6. In a particular list of three-digit perfect squares, the first perfect square can be turned into each of the others by rearranging its digits. What is the largest number of distinct perfect squares that could be in the list?

Always know your perfect squares up to 1000.
121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 784, 841, 900, 961
144 has 441.
169 has 196 and 961. That's pretty good.
256 only has 625. It's gotta be at least 3. But I don't see any more.
3 **Ans.**

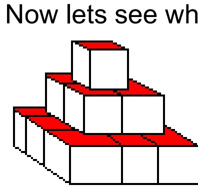
7. Charlie builds a square pyramid-like figure using unit cubes. The first level has one cube. There will be 8 levels. On each level, the vertices of the largest bottom square coincide with the centers of the top faces of the four corner cubes of the level below. We are asked to find the surface area of the resulting solid.
The first 3 levels look (sort of) like this:



The area of each unit cube is 1.
Looking at everything that is vertically visible we see the following:



Everything in red is part of this section of the area. Don't forget those we can't see in the back. So in the front we have 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36. The same is true on each side and in the back.
36 × 4 = 144
Now lets see what's visible on top.



This is a little harder.
On the first level we have a whole square. On the second level, we have 4 corner squares, each of

which lose $\frac{1}{4}$ of its area to the square sitting on top.

$\frac{3}{4} \times 4 = 3$

The third level has the same four corner squares each of which lose $\frac{1}{4}$ of its area to the squares on top and 4 more squares each of which lose $\frac{1}{2}$ of its area to the squares on top. Thus the third level area is:

$3 + 4 \times \frac{1}{2} = 5$

The fourth level will have 4 more of these squares for an area of 7. Aha! A pattern. The total surface area for on top is 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 = 16 × 4 = 64

Now, what's showing on the bottom? Just the bottom of the eight level is showing. That's 8^2 squares or 64.

144 + 64 + 64 = 272 **Ans.**

8. $r + \frac{1}{r} = 2007 + \frac{1}{2007}$

$s + \frac{1}{s} = 2006 + \frac{1}{2006}$

$\frac{r^2 + 1}{r} = \frac{4028050}{2007}$

$\frac{s^2 + 1}{s} = \frac{4024037}{2006}$

$2007r^2 + 2007 = 4028050r$

$2007r^2 – 4028050r + 2007 = 0$

(2007r – 1)(r – 2007) = 0

r = 2007 and

r = $\frac{1}{2007}$

$2006s^2 – 4024037s + 2006 = 0$

(2006s – 1)(s – 2006) = 0

s = 2006 and

s = $\frac{1}{2006}$

To maximize r – s, take the largest r and the smallest s.

r = 2007

s = $\frac{1}{2006}$

$2007 – \frac{1}{2006} = 2006\frac{2005}{2006}$ **Ans.**

9. For positive integers K and T, it is true that $27 \times K = 3^{T+2}$

So what is the smallest possible value of K ÷ T?

$27 \times K = 3^3 \times K = 3^{T+2}$, K > 100 and K ÷ T is an integer.

K is obviously a power of 3.

T must be divisible by 3 if K ÷ T is an integer. So what happens if T = 3?

Then $3^{T+2} = 3^5$ but K would be 3^2 which violates the requirement that K > 100.

How about T = 6?

Then $3^{T+2} = 3^8$ and K would be 3^5. While that's greater than 100, it's not divisible by 2 and 6 is so K ÷ T wouldn't be an integer. So, how about T = 9? Then $3^{T+2} = 3^{11}$ and K = 3^8

$\frac{3^8}{9} = \frac{3^8}{3^2} = 3^6 = 729$ **Ans.**

10. There are exactly two blue marbles in a bag of three marbles. If Kia randomly chooses two marbles without replacement, the probability of choosing the two blue marbles is one-third. Now suppose Kia adds more marbles to the bag. Then the probability of picking two blue marbles without replacement is still one-third. We are asked to find the least number of marbles that could be in the bag after the additional marbles have been added.

Let's start at the beginning. If there are 3 marbles and 2 are blue, then the probability of choosing 2 blue ones is:

$\frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$

Yes, that's right.

Let x be the total number of marbles after all the other marbles are added and let y be the total number of blue marbles. Then we know that:

$$\frac{y}{x} \times \frac{y-1}{x-1} = \frac{1}{3}$$

3 × y × (y-1) = x × (x – 1)
This means that the numerator and denominator of $\frac{y}{x} \times \frac{y-1}{x-1}$ must each factor into two factors that differ by one and each must be divisible by 3.
Certainly

$\frac{2}{3} \times \frac{1}{2}$ but we already knew that.

What about a denominator of 3 × 4? This wouldn't work since the numerator would be 4 $\left(\frac{4}{12} = \frac{1}{3}\right)$ which can't be factored into two factors that differ by 1.
We can't use a denominator of 4 × 5 because it's not divisble by 3.
What about 5 × 6?
The numerator would have to be 10 and that doesn't satisfy the requirements.
Same is true for 6 × 7.
What about 8 × 9? Also true because the denominator would have to be 24.
So how about 9 × 10. The numerator would be 30 and that can be factored into 6 × 5 and we have a winner. 10 **Ans.**