MATHCOUNTS

2013 State Competition Solutions

Are you wondering how we could have possibly thought that a Mathlete® would be able to answer a particular Sprint Round problem without a calculator?

Are you wondering how we could have possibly thought that a Mathlete would be able to answer a particular Target Round problem in less 3 minutes?

Are you wondering how we could have possibly thought that a Mathlete would be able to answer a particular Team Round problem with less than 10 sheets of scratch paper?

The following pages provide solutions to the Sprint, Target and Team Rounds of the 2013 MATHCOUNTS® State Competition. Though these solutions provide creative and concise ways of solving the problems from the competition, there are certainly numerous other solutions that also lead to the correct answer, and may even be more creative or more concise! We encourage you to find numerous solutions and representations for these MATHCOUNTS problems.

Special thanks to volunteer author Mady Bauer for sharing these solutions with us and the rest of the MATHCOUNTS community!

2013 State Competition

Sprint Round

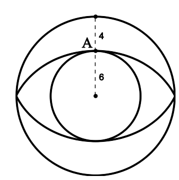
1. The sum of 2 numbers is 4. Their difference is 2. What is their product?
   Let x and y be the two numbers.
   $x + y = 4$
   $x - y = 2$
   $2x = 4 + 2 = 6$
   $x = 3$
   $3 + y = 4$
   $y = 1$
   $3 \times 1 = 3$ **Ans.**

2. Mary and Ann ride their bikes to meet somewhere between their two houses. At 11 a.m. Mary has traveled half the distance between the two houses. Ann has only covered 3/8 of the distance. They are still one mile apart. What is the distance between their two houses?
   Let d represent the distance between their two houses. Then we have $d = (1/2)d + (3/8)d + 1$. Solving for d yields $d = (7/8)d + 1 \rightarrow (1/8)d = 1 \rightarrow$ $d = 8$ **Ans.**

3. A bag contains 8 blue marbles, 4 red marbles and 3 green marbles. What is the probability of not drawing a green marble?
   There are a total of 8 + 4 + 3 = 15 marbles. Of those, 8 + 4 = 12 marbles are not green. That's a probability of 12/15 = 4/5 **Ans.**

4. The arithmetic mean of 11 numbers is 78. If 1 is subtracted from the first number, 2 from the second, and so on, what is the arithmetic mean of the resulting numbers?
   With a mean of 78, the sum of the 11 original numbers is $11 \times 78 = 858$. The total amount being subtracted from this sum is $1 + 2 + 3 + \ldots + 9 + 10 + 11 = 12 \times (11/2) = 66$. The sum of the 11 numbers that result after subtracting 66 is $858 - 66 = 792$, and their mean is $792 \div 11 = 72$ **Ans.**

5. The logo contains a center circle with area 36π in^2. The circle is tangent to each crescent at its widest point (A and B). The shortest distance from A to the outer circle is 1/3 the diameter of the smaller circle. What is the area of the larger circle?
   First, let's find the radius of the smaller circle. We know that the area is 36π, so $\pi r^2 = 36\pi$, and $r = 6$ in. The diameter, then, is 12 in. If follows that shortest distance from A to the outer circle, which is 1/3 the diameter of the smaller circle, is $(1/3) \times 12 = 4$ in.



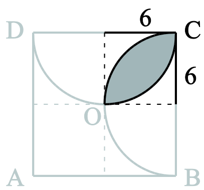
   That means the radius of the larger circle is 6 + 4 = 10 in, and its area, in square inches, is $(10^2)\pi = 100\pi$ **Ans.**

6. Doris buys a DVD originally priced at \$12 and now on sale for 20% off. How much will Doris save?
   20% of 12 is $0.20 \times 12 = \$2.40$ **Ans.**

7. What is the value of $\frac{444^2 - 111^2}{444 - 111}$?
   $444^2 - 111^2 = (444 + 111)(444 - 111)$,
   so we have $\frac{g}{r}$ = 444 + 111 = 555 **Ans.**

8. The digits of positive integer n have a product of 20 and a sum of 13. What is the smallest possible value of n?
First, let's determine the digits of n, then we can arrange them to yield the smallest value.
Suppose n is a 2-digit integer, AB, where A is the tens digit and B is the units digit. We need $A \times B = 20$ and $A + B = 13$. The only pair of 1-digit integers with a product of 20 is 5×4, but $5 + 4 \neq 13$. So, n has more than 2 digits. Now suppose that n is a 3-digit integer, ABC. Again, we need $A \times B \times C = 20$ and $A + B + C = 13$. The 1-digit triples with a product of 20 are $5 \times 4 \times 1$, but $5 + 4 + 1 \neq 13$, and $5 \times 2 \times 2$, but $5 + 2 + 2 \neq 13$. So, n must have more than 3 digits. Let's now assume n is a 4-digit integer, ABCD. Again, we need $A \times B \times C \times D = 20$ and $A + B + C + D = 13$. There are two groups of 4 integers with a product of 20: $5 \times 4 \times 1 \times 1$, but $5 + 4 + 1 + 1 \neq 13$, and $5 \times 2 \times 2 \times 1$, but $5 + 2 + 2 + 1 \neq 13$. Notice that $5 + 4 + 1 + 1 = 11$ is just 2 less than 13, and $5 + 2 + 2 + 1 = 10$ is 3 less than 13. Adding 2 more 1s to the first sum will yield a 6-digit integer, while adding 3 more 1s to the second sum yields a 7-digit integer. We want the smallest integer, so $n = 111145$ **<u>Ans.</u>**

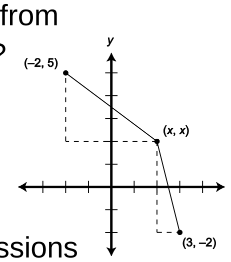
9. In square ABCD BC = 12. BOC and DOC are semicircles. What is the area of the shaded region?



We know the two semicircles intersect at the center of square ABCD. We can divide it into four smaller squares, as shown. Focusing on the smaller square with a vertex at C, notice that the area of each of the two *unshaded* regions is the difference between the area of the smaller square and the area of one quarter of a circle with radius 6 cm (since the diameter BC = 12 cm). With side length 6 cm, the area of this smaller square is 36 cm^2. The area of the quarter-circle is $(1/4)\pi(6^2) = 9\pi$ cm^2, and the area of the two unshaded regions is $2(36 - 9\pi) = 72 - 18\pi$ cm^2. So, in square centimeters, the area of the shaded region is $36 - (72 - 18\pi) = -36 + 18\pi$ or $18\pi - 36$ **<u>Ans.</u>**

10. Real numbers a and b satisfy $b = \frac{2a-4}{5} + \frac{3a+1}{5}$. What is $a - b$?
Let's manipulate the equation to isolate the expression $a - b$ on one side:
$2a - 4 + 3a + 1 = 5b \rightarrow 5a - 3 = 5b \rightarrow 5a - 5b = 3 \rightarrow a - b = 3/5$ **<u>Ans.</u>**

11. The point (x, x) is equidistant from $(-2, 5)$ and $(3, -2)$. What is x?



As shown, the distances from (x, x) to $(-2, 5)$ and to $(3, -2)$ are the lengths of the hypotenuses of the two right triangles. We can write expressions for these distances using the Pythagorean Theorem. Then we set them equal to each other and solve:

$$(x + 2)^2 + (3 - x)^2 = (5 - x)^2 + (x + 2)^2$$
$$(3 - x)^2 = (5 - x)^2$$
$$9 - 6x + x^2 = 25 - 10x + x^2$$
$$9 - 6x = 25 - 10x$$
$$4x = 16$$
$$x = 4$$ **<u>Ans.</u>**

12. In a bag of marbles, 2/5 are red, 3/10 are white marbles and 1/10 are blue. The remaining 10 marbles are green. How many marbles are in the bag?
We know that $(2/5) + (3/10) + (1/10) = 8/10 = 4/5$ of the marbles in the bag are not green. Then the10 green marbles

represent 1/5 of the marbles in the bag. It follows that the number of marbles in the bag is $10 \times 5 = 50$ **Ans.**

13. t is 40% greater than p. p is 40% less than 600. What is $t - p$?
    40% less than 600 is the same as 60% of 600. So $p = 0.6 \times 600 = 360$. 10% of 360 is $0.1 \times 360 = 36$. So 40% of 360 = $4 \times$ 10% of 360 = $4 \times 36 = 144$. Therefore, t, which is 40% greater than p is $360 + 144 = 504$. It follows that $t - p = 504 - 360 = 144$ **Ans.**

14. How many ways can all six numbers in the set {4, 3, 2, 12, 1, 6} be ordered so that a comes before b whenever a is a divisor of b?
    Since 1 is a divisor of all the other numbers, it must always be first. Since all the other numbers are divisors of 12, it must always be last. For the remaining numbers we know that 2 must be before 4 and 6, but can be after 3; 3 must be before 6, but can be either before or after 2 and 4; 4 must be after 2, but can be before or after 3 and 6. The following orders are possible: 1, 2, 3, 4, 6, 12; 1, 2, 3, 6, 4, 12; 1, 2, 4, 3, 6, 12; 1, 3, 2, 4, 6, 12; 1, 3, 2, 6, 4, 12. The number of ways to order the numbers is 5 **Ans.**

15. What is the units digit of the product $7^{23} \times 8^{105} \times 3^{18}$?
    First let's look at the pattern of powers of 7. We have $7^1 = 7$, $7^2 = 49$, $7^3 = 343$, $7^4 = 2401$ and $7^5 = 16{,}807$. So the units digits 7, 9, 3 and 1 repeat every four powers of 7. In other words, when 7 is raised to a power for which the remainder is 1 when divided by 4, the result will end in 7. When raised to a power that yields a remainder of 2 when divided by 4, the result will end in 9, and so on. Since 23/4 = 5 r3, the units digit of 7^{23} will be 3.
    Now look at the pattern of powers of 8. We have $8^1 = 8$, $8^2 = 64$, $8^3 = 512$, $8^4 = 4096$ and $8^5 = 32{,}768$. So the units digits 8, 4, 2 and 6 and will repeat every four powers of 8. Since 105/4 = 26 r1, the units digit of 8^{105} will be 8.
    Finally, let's find the pattern for powers of 3. We have $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$ and $3^5 = 243$. The units digits 3, 9, 7 and 1 will repeat every four powers of 3. Since 18/4 = 4 r2, 3^{18} will have a units digit of 9. Multiplying $3 \times 8 \times 9 = 216$, we conclude that the units digits of the product $7^{23} \times 8^{105} \times 3^{18}$ will be 6 **Ans.**

16. If $4(a - 3) - 2(b + 5) =$ and $5b - a = 0$, what is $a + b$?
    Rewriting the second equation, we see that $a = 5b$. When we substitute this into the first equation and solve, the result is
    $$4(5b - 3) - 2(b + 5) = 14$$
    $$20b - 12 - 2b - 10 = 14$$
    $$18b - 22 = 14$$
    $$18b = 36$$
    $$b = 2$$
    $$a = 5 \times 2 = 10$$
    So, $a + b = 10 + 2 = 12$ **Ans.**

17. The two cones have parallel bases and common apex T. TW = 32 m, WV = 8 m and ZY = 5 m. What is the volume of the frustum with circles W and Z as its bases?
    The cone has height 32 m and the radius of the base is 8 m. The volume of the larger cone is $(1/3)\pi r^2 h = (1/3)\pi(8^2)(32) = (2048/3)\pi$ m^3. If we find the volume of the smaller cone, we can subtract that value from the volume of the larger cone to get the volume of the frustum. We know the radius of the smaller cone's base is 5 m, but we need

to find its height. Triangle TXY is similar to Triangle TUV, so ZY/WV = TZ/TW → 5/8 = TZ/32 → TZ = (32 × 5)/8 = 20 m. The volume of the smaller cone, then, is $(1/3)\pi(5^2)(20) = (500/3)\pi\ m^3$. It follows that the area of the frustum, in cubic meters, is (2048/3)π − (500/3)π = (1548/3)π = 516π **Ans.**

18. A coin is flipped until it either lands heads two times or tails two times but not necessarily in a row. If the first flip lands heads, what is the probability that a second head occurs before two tails? Let's just list the ways this can occur: HH, HTH. Anything more either gives us two tails or more than two heads. The probability of getting one more H is 1/2. The probability of getting a TH is (1/2) × (1/2) = 1/4. Therefore, the probability of getting HH or HTH is (1/2) + (1/4) = 3/4 **Ans.**

19. The product of two consecutive integers is five more than their sum. What is the smallest possible sum of two such consecutive integers?
Let x represent the first integer. We can write the following equation:
$x(x + 1) = x + x + 1 + 5$
$x^2 + x = 2x + 6$
$x^2 - x - 6 = 0$
$(x - 3)(x + 2) = 0$
$x = 3$ or $x = -2$
If $x = 3$ then $x + x + 1 = 3 + 4 = 7$
If $x = -2$ then $x + x + 1 = -2 - 1 = -3$ **Ans.**

20. 4 nickels, 1 penny and 1 dime are divided among 3 piggy banks. Each bank received two coins. Labels indicating the amount in each bank were made. These are 6¢, 10¢ and 15¢. The wrong labels were put on each bank. There was at least one penny in the bank labeled as 15¢. What was the actual combined value of the two coins contained in the piggy bank that was labeled 6¢?
Let's first look at the combinations we need to make 6¢, 10¢ and 15¢:
6 = 5 + 1
10 = 5 + 5
15 = 10 + 5
(Remember, we need two coins in each bank.) The bank marked as 15¢ has a penny in it. Since we know the total value of the coins inside the bank does not match its label, it must also contain a nickel. That leaves the banks labeled 10¢ and 6¢. Since they too are labeled incorrectly, the bank labeled 10¢ must contain a dime and a nickel, and the bank labeled 6¢ must contain two nickels, with a combined value of 10 **Ans.**

21. A 9 × 9 multiplication grid was filled in completely. What is the sum of the 81 products.
Written out, the sum is:
1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 +
2 × (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) +
3 × (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) +
… +
9 × (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9)
You could multiple everything and add but that is time consuming. Let $x = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 \rightarrow x = 45$
We can rewrite the original sum as $x + 2x + 3x + \ldots + 9x = 45x$. The sum of the products is 45 × 45 = 2025 **Ans.**

22. How many seven-letter words can be made from the word MAKAALA if each consonant must be followed by a vowel? Let's start having only one A follow a consonant.

Consider permutations of A__A__A__A, where the blanks are filled with the consonants M, K and L. There are 6 such permutations. Similarly, consider permutations of __AA__A__A, __A__AA__A and __A__A__AA, where the blanks are filled with the M, K and L. Again, there are 6 permutations of each. Permutations with 3 adjacent As won't work. Therefore, number of permutations that meet the stated criteria is $6 \times 4 = 24$ **Ans.**

23. Given $f(x) = 3x^2$, what is the x-coordinate of the point of intersection of the graphs of $y = f(x)$ and $y = f(x - 4)$?
These two functions intersect when $f(x) = f(x - 4)$. Substituting and solving for x, we have

$$3x^2 = 3(x - 4)^2$$
$$3x^2 = 3(x^2 - 8x + 16)$$
$$x^2 = x^2 - 8x + 16$$
$$8x - 16 = 0$$
$$8x = 16$$
$$x = 2$$ **Ans.**

24. In isosceles trapezoid ABCD, AB = 4, CD = 10. Points E and EF are on CD and BE is parallel to AD. AF is parallel to BC. AF and BE intersect at point G. What is the ratio of the area of triangle EFG to the area of trapezoid ABCD?
Given that segments BE and AD are parallel, it follows that ABED is a parallelogram, with DE = 4. Similarly, since segments AF and BC are parallel, it follows that ABCE is a parallelogram, with FC = 4. Therefore, EF = 10 − 8 = 2. Notice that $\triangle$ABG ~ $\triangle$EFG (Angle-Angle), and the ratio EF/AB = 2/4 = 1/2. Since these two triangles are similar, if we let x represent the height of $\triangle$EFG, the height of $\triangle$ABG must be $2x$. Therefore, the height of the trapezoid is $x + 2x = 3x$. Recall, the area of a trapezoid is given by $(1/2) \times \text{height} \times (\text{base}_1 + \text{base}_2)$. So the area of trapezoid ABCD is $(1/2) \times 3x \times (4 + 10) = 21x$. The area of $\triangle$EFG is $(1/2) \times 2 \times x = x$. Therefore, the ratio of the area of $\triangle$EFG to the area of the trapezoid ABCD is $x/(21x) = 1/21$ **Ans.**

25. The sum of five consecutive positive even integers is a perfect square. What is the smallest possible integer that could be the least of the five integers?
Let the five consecutive even integers be represented by $(n - 4)$, $(n - 2)$, n, $(n + 2)$, $(n + 4)$, with sum $n - 4 + n - 2 + n + n + 2 + n + 4 = 5x$. Since we are told that the sum is a perfect square, we need to find a perfect square that is a multiple of 5. Start with 25, in which case we have $5x = 25 \rightarrow x = 5$. Since 5 is not even, we move on to the next perfect square that is a multiple of 5: 100. We have $5x = 100 \rightarrow x = 20$, and the smallest of the five consecutive even integers is $x - 4 = 16$ **Ans.**

26. If $12_3 + 12_5 + 12_7 + 12_9 + 12_x = 101110_2$, what is the value of x?
Let's begin by finding the value of each term in base 10. We have
$12_3 = (1 \times 3^1) + (2 \times 3^0) = 3 + 2 = 5$
$12_5 = (1 \times 5^1) + (2 \times 5^0) = 5 + 2 = 7$
$12_7 = (1 \times 7^1) + (2 \times 7^0) = 7 + 2 = 9$
$12_9 = (1 \times 9^1) + (2 \times 9^0) = 9 + 2 = 11$
$12_x = (1 \times x^1) + (2 \times x^0) = x + 2$
$101110_2 = (1 \times 2^5) + (1 \times 2^3) + (1 \times 2^2) + (1 \times 2^1) = 32 + 8 + 4 + 2 = 46$
Now we can rewrite the equation and solve:
$5 + 7 + 9 + 11 + x + 2 = 46$
$34 + x = 46$
$x = 12$ **Ans.**

27. A box contains r red balls and g green balls. When r more red balls are added, the probability of drawing a red ball increases by 25%. Find the probability of randomly drawing a red ball from the box before the extra red balls were added.

The probability of drawing a red ball before more red balls are added is $\frac{r}{g+r}$. The probability of drawing a red ball after more red balls are added is $\frac{2r}{g+2r}$, which is 25% larger than $\frac{r}{g+r}$. We can write the equation

$$\frac{2r}{g+2r} = \frac{5}{4} \times \frac{r}{g+r}$$

$$\frac{2r}{g+2r} = \frac{5r}{4(g+r)}$$

$$2r \times 4(g+r) = 5r \times (g+2r)$$

$$8rg + 8r^2 = 5rg + 10r^2$$

$$3rg = 2r^2$$

$$3g = 2r$$

$$\frac{g}{r} = \frac{2}{3}$$

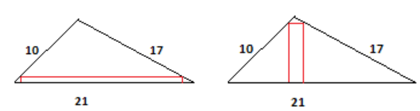
So the ratio of green balls to red balls in the box is 2/3. Thus, the probability of randomly drawing a red ball is 3/5 **Ans.**

28. Connex contains one 4-unit piece, two 3-unit pieces, three 2-unit pieces and four 1-unit pieces. How many different 10-unit pieces (where order matters) can be arranged?

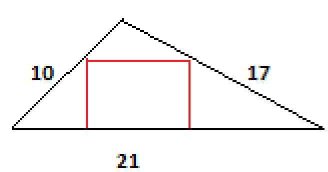
Let's start by creating 10-unit pieces using the 4-unit piece. Our arrangements are:
4-3-3 (3 permutations)
4-3-2-1 (4! = 24 permutations)
4-3-1-1-1 ($5 \times [4!/(3!1!)] = 5 \times 4 = 20$ permutations)
4-2-2-2 (4 permutations)
4-2-2-1-1 ($5 \times [4!/(2!2!)] = 5 \times 6 = 30$ permutations)
4-2-1-1-1-1 ($6 \times [5!/(4!1!)] = 6 \times 5 = 30$ permutations)
Let's look at arrangements using one or more 3-unit pieces and no 4-unit piece:
3-3-2-2 (4!/(2!2!)
3-3-2-1-1 ($5 \times (4!/(2!2!) = 5 \times 6 = 30$ permutations)
3-3-1-1-1-1 (6!/(4!2!) = permutations)
3-2-2-2-1 ($5 \times 4!/(3!1!) = 5 \times 4 = 20$ permutations)
3-2-2-1-1-1 ($6 \times 5!/(3!2!) = 6 \times 10 = 60$ permutations)
Finally we'll consider arrangements using only 1- and 2-unit pieces:
2-2-2-1-1-1-1 (7!/(4!3!) = 35 permutations)
Let's add them all up:
(3 + 24 + 20 + 4) + (30 + 30 + 6 + 30) + (15 + 20 + 60 + 35) =
51 + 96 + 130 = 277 **Ans.**

29. What is the largest possible area a rectangle inscribed in this triangle can have?

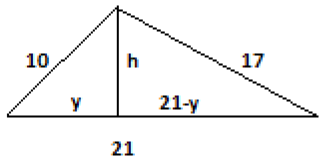
We could look at very wide rectangles or we could look at very high rectangles.



But to maximize the area means to find the rectangle that is as close to being a square as possible.



First, let's figure out the height of the triangle.



For the two right triangles, we can write the following equations using the Pythagorean Theorem:

$h^2 + y^2 = 10^2$ [Eq. 1]

$h^2 + y^2 = 100$

$h^2 + (21 - y)^2 = 17^2$ [Eq. 2]

$h^2 + 441 - 42y + y^2 = 289$

Substituting for $h^2 + y^2$, we get

$100 + 441 - 42y = 289$

$42y = 252$

$y = 6$

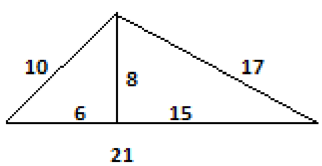
Substituting this for y in Eq. 1 and solving for h, we get

$h^2 + 6^2 = 100$

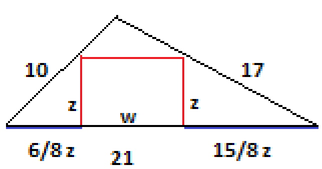
$h^2 = 64$

$h = 8$

So, we know the following measurements for the triangle.



The ratio of y to h is 6/8. The ratio of $21 - y$ to h is 15/8.



Back to the rectangle, let w and z represent our rectangle's width and height, respectively. The triangle created on the left side is similar to the 6-8-10 triangle from the previous figure. So if the height is z, the side corresponding to 6 has length $(6/8)z$. Similarly, on the right, we have a triangle similar to the 8-15-17 right triangle, and the side corresponding to 15 has length $(15/8)z$. We can write the equation:

$(6/8)z + w + (15/8)z = 21$

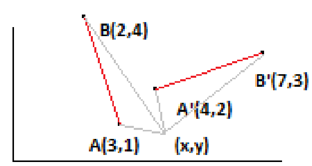
$(21/8)z + w = 168/8$

$w = (168/8) - (21/8)z.$

Now suppose $z = 8/2 = 4$. It follows that $w = (168/8) - ((21/8) \times 4) = (168/8) - 84/8 = 84/8$. The rectangle has an area of $wz = 4 \times (84/8) = 84/2 = 42$. When we check for $z = 3$ and $z = 5$, we get an area of 315/8, which is less than the maximum area of 42. **Ans.**

Notice that the area of the original triangle is (1/2) × 21 × 8 = 84, which is twice the area of the rectangle. In fact, the maximum area of a rectangle inscribed in a triangle is always half the area of the triangle.

30. A line segments with endpoints A(3, 1) and B(2, 4) is rotated about a point in the plane so that its endpoints are moved to A'(4, 2) and B'(7, 3). What are the coordinates of the center of rotation?



Let the point (x, y) be the center of rotation. Then the distance between (x, y) and (3, 1) is the same as the distance between (x, y) and (4, 2). So we have

$(x - 3)^2 + (y - 1)^2 = (x - 4)^2 + (y - 2)^2$

$x^2 - 6x + 9 + y^2 - 2y + 1 = x^2 - 8x + 16 + y^2 - 4y + 4$

$-6x + 9 - 2y + 1 = -8x + 16 - 4y + 4$

$-6x - 2y + 10 = -8x - 4y + 20$

$2x + 2y = 10$

$x + y = 5$ (Eq. 1)

Similarly, the distance between (x, y) and (2, 4) is the same as the distance between (x, y) and (7, 3).

$$(x-2)^2+(y-4)^2=(x-7)^2+(y-3)^2$$
$$x^2-4x+4+y^2-8y+16=$$
$$x^2-14x+49+y^2-6y+9$$
$$-4x+4-8y+16=-14x+49-6y+9$$
$$-4x-8y+20=-14x-6y+58$$
$$10x-2y=38$$
$$5x-y=19 \quad \text{(Eq. 2)}$$

Adding Eq. 1 and Eq. 2, we have:

$$x+y+5x-y=5+19$$
$$6x=24$$
$$x=4$$

Substituting this into Eq. 1, we get

$$4+y=5$$
$$y=1$$

Thus, the ordered pair of the center of rotation is (4,1) **Ans.**

Target Round

1. Jonas scored 88 points prior to the last game. He scored 23 points in the last game. This made his season average 18.5 points per game. Find the number of games he played.
   88 + 23 = 111
   Let x represent the number of games played. We can write $111/x = 18.5 \rightarrow x = 111/18.5 \rightarrow x = 6$ **Ans.**

2. The dart board contains 20 uniquely numbered sectors. Malaika hits a number she aims for only half the time. The other half, she randomly is an adjacent number on either side. Malaika aims for the same number for 20 throws. To get the highest possible score, which number should she aim for?
   The numbers are in the following order from the top and going clockwise:
   20, 1, 18, 4, 13, 6, 10, 15, 2, 17, 3, 19, 7, 16, 8, 11, 14, 9, 12, 5
   It may be ugly but there are several minutes to solve this – so…
   Let's just look at 20, 1, 18.
   When Malaika aims for the 1 she will hit it 10 of the 20 times; she will hit 20 only 5 times and 18 only 5 times. That's a score of $(1 \times 10) + (20 \times 5) + (18 \times 5) = 200$. We can generalize this by writing $10b + 5(a + c)$, where $a = 20$, $b = 1$ and $c = 18$. Let's create a table the possible values:

b	***a + c***	***10b***	***5(a+c)***	***Total***
20	6	200	30	230
1	38	10	190	200
18	5	180	25	205
4	31	40	155	195
13	10	130	50	180
6	23	60	115	175
10	21	100	105	205
15	12	150	60	210
2	32	20	160	180
17	5	170	25	195
3	36	30	180	210
19	10	190	50	240
7	35	70	175	245
16	15	160	75	235
8	27	80	135	215
11	22	110	110	220
14	20	140	100	240
9	26	90	130	220
12	14	120	70	190
5	32	50	160	210

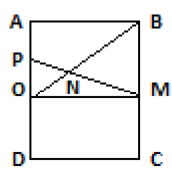
   To get the highest possible score, Malaika should aim for the 7 **Ans.**

3. Joy rides her bike up the hill passing Greg after 3 km. Greg is walking down at 1 m/s. Joy rides another 7 km up the hill and then doubles her speed going down the hill. Joy and Greg arrive at Joy's starting point at the same time. What is Joy's average speed going downhill?
   Greg walked downhill for 3 km at 1 m/s. 3 km = 3000 m. It would take Greg 3000 seconds to travel 3000 m.
   Let s represent the average speed that

Joy was going uphill, in meters per second. Joy rode 7 km, or 7000 m, at this speed and then 7 + 3 = 10 km, or 10,000 m, down at twice the speed or 2*s*. Joy did this in 3000 seconds. We can write the following equation:
$(7000/s) + (10{,}000/2s) = 3000 \rightarrow$
$(7000/s) + (5000/s) = 3000 \rightarrow$
$12{,}000/s = 3000 \rightarrow 3000s = 12{,}000 \rightarrow$
$s = 4$ and $2s = 2 \times 4 = 8$ **Ans.**

4. In rectangle ABCD, BC = 2AB, points O and M are midpoints of segments AD and BC, respectively. Point P bisects segment AO, and OB = 6√2. What is the area of △NOP?

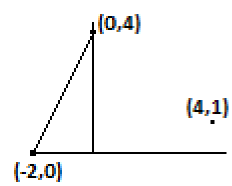
   Since BC = AD = 2AB and O is the midpoint of side AD, it follows that AO = AB. Let *x* represent the length of segment AO. Using the Pythagorean Theorem, we can write $x^2 + x^2 = (6\sqrt{2})^2 \rightarrow 2x^2 = 72 \rightarrow x^2 = 36 \rightarrow x = 6$. So AO = AB = 6. Since P bisects AO, it follows that PO = 3. Constructing segment OM parallel to side AB, as shown, creates square, ABMO. The area of triangle MOP is $(1/2) \times 3 \times 6 = 9$. The area of triangle MOB is $(1/2) \times 6 \times 6 = 18$. By the Angle-Angle Postulate, we know that triangles NOP and NBM are similar. We also know that PO = 3 and BM = 6, so the sides of triangle MOB are twice the sides of triangle MOP.



   Let *h* represent the height of triangle MOB. Then the area of MOB is $(1/2) \times h \times 6 = 3h$. Since the height of triangle MOP is $(1/2)h$, it follows that its area is $(1/2) \times (1/2)h \times 3 = (3/4)h$. Thus, the area of triangle MNB is 4 times the area of triangle NOP.
   So if *x* is the area of triangle NOP, and *y* is the area of triangle ONM, then the area of triangle MOP is the sum of the areas of triangles NOP and ONM. In other words, $x + y = 9$. The area of triangle MOB is the area of triangle MNB plus the area of triangle ONM, so $4x + y = 18 \rightarrow 3x = 9 \rightarrow x = 3$ **Ans.**

5. If $40q = p + (p/3) + (p/9) + (p/27)$, what is the ratio q/p?
   Let's begin by simplifying the equation and isolating one of the variables:
   $40q = p + (p/3) + (p/9) + (p/27)$
   $40q = (27p + 9p + 3p + p)/27$
   $40q = 40p/27$
   $q = p/27$
   $q/p = 1/27$ **Ans.**

6. What is the length of the shortest segment that can be drawn from the point (4, 1) to $2x - y + 4 = 0$?
   $2x - y + 4 = 0$
   $y = 2x + 4$
   Let's just plot it to see what it looks like.



   (−2, 0) and (0, 4) are points on the graph of $y = 2x + 4$
   The shortest segment between (4, 1) and the line will be on a line perpendicular to $y = 2x + 4$
   The slope, m, of the line is 2.
   The slope for the perpendicular line is
   $$-\frac{1}{m} = -\frac{1}{2}$$
   Thus, the equation for the perpendicular line is
   $$y = -\frac{1}{2} + b$$
   (4, 1) is a point on this line.
   $$1 = -\frac{1}{2} \times 4 + b = -2 + b$$

$b = 3$

So the equation is $y = -\frac{1}{2}x + 3$

The point that we need to find is when

$-\frac{1}{2}x + 3 = 2x + 4$

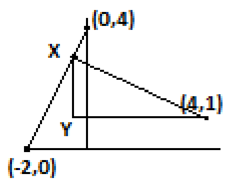
$-\frac{1}{2}x = 2x + 1$

$\frac{5}{2}x = -1$

$x = -\frac{2}{5}$

$y = -\frac{1}{2} \times \frac{2}{5} + 3 = 3\frac{1}{5}$

Therefore, we must determine the length of the line connecting (4, 1) and $\left(-\frac{2}{5}, 3\frac{1}{5}\right)$.



Let X be the point $\left(-\frac{2}{5}, 3\frac{1}{5}\right)$

Then Y is the point $\left(-\frac{2}{5}, 1\right)$

The length of XY is

$\frac{16}{5} - \frac{5}{5} = \frac{11}{5}$

The length of Y to (4, 1) is:

$4 - \left(-\frac{2}{5}\right) = 4\frac{2}{5}$

The length from X to (4,1) is:

$\sqrt{\left(\frac{11}{5}\right)^2 + \left(\frac{22}{5}\right)^2} =$

$\sqrt{4.84 + 19.36} = \sqrt{24.2} \approx 4.9193495 \approx$ 4.92 **Ans.**

7. How many positive two-digit integers have exactly 8 positive factors?

1 and the number itself are 2 factors. There are 6 other factors. We could look at every two-digit integer and brute force it but let's see if there isn't a simple way to figure it out.

Let a, b, and c be the three factors.

If a, b, and c are prime then the factors would be:

1, a, b, c, ab, bc, ab, abc

1: 2 × 3 × 5 = **30**

30 = 1 × 30, 2 × 15, 3 × 10 and 5 × 6.

2: 2 × 3 × 7 = **42**

42 = 1 × 42, 2 × 21, 3 × 14 and 6 × 7

3: 2 × 3 × 11 = **66**

66 = 1 × 66, 2 × 33, 3 × 66 and 6 × 11

4: 2 × 3 × 13 = **78**

78 = 1 × 78, 2 × 39, 3 × 26 and 6 × 13

5: 2 × 3 × 17 = 102 so

2 × 5 × 7 = **70**

70 = 1 × 70, 2 × 35, 5 × 14 and 7 × 10.

6: 2 × 5 × 11 = 110

2 × 7 × 11 = 140

3 × 5 × 7 = 105

We are done with primes. Now let's look at relatively prime. Let's try 3 factors of the form: a, a^2, b so the other three factors are a^3, a^2b and ab

2 × 4 × 3 = **24**

24 = 1 × 24, 2 × 12, 3 × 8 and 4 × 6

7: 2 × 4 × 5 = **40**

40 = 1 × 40, 2 × 20, 4 × 10 and 5 × 8

8: 2 × 4 × 7 = **56**

56 = 1 × 56, 2 × 28, 4 × 14 and 7 × 8

9: 2 × 4 × 11 = **88**

88 = 1 × 88, 2 × 44, 4 × 22 and 8 × 11

10: 2 × 4 × 13 = 104

So let's deal with 3.

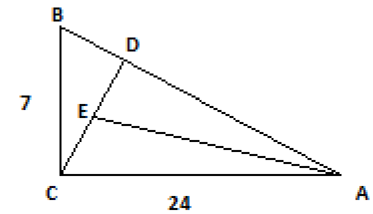
3 × 9 × 2 = **54**

54 = 1 × 54, 2 × 27, 3 × 18 and 6 × 9

At this point, no matter what other combination we try, we'll be over 100.

10 **Ans.**

8. In triangle ABC, AC = 24 and BC = 7. CD is perpendicular to AB. The bisector of the smallest angle of triangle ABC intersects CD at point E. Find the length of ED.



Given that AC = 24 and BC = 7, that means that BA must be 25.
Let x = the length of BD.
Then the length of AD = 25 − x.
Let h = the length of DC.
We can create two equations.

$h^2 + x^2 = 7^2 = 49$
$h^2 + (25 - x)^2 = 24^2 = 576$
$h^2 = 49 - x^2$
$h^2 = 576 - (25 - x)^2$
$49 - x^2 = 576 - (25 - x)^2$
$49 - x^2 = 576 - (625 - 50x + x^2)$
$49 - x^2 = 576 - 625 + 50x - x^2$
$49 = -49 + 50x$
$50x = 98$
$x = \frac{98}{50} = \frac{49}{25}$
$h^2 + x^2 = 49$
$h^2 + \left(\frac{49}{25}\right)^2 = 49 = \frac{1225}{25}$
$h^2 + \frac{2401}{625} = \frac{30625}{625}$
$h^2 = \frac{30625}{625} - \frac{2401}{625} = \frac{28224}{625}$
$h = \sqrt{\frac{28224}{625}} = \sqrt{45.1584} = 6.72 = 6\frac{18}{25}$

The length of AD is $25 - \frac{49}{25} = 23\frac{1}{25}$.
Now, from the bisector theorem:

$\frac{DE}{EC} = \frac{AD}{AC} = \frac{23\frac{1}{25}}{24} = \frac{\frac{576}{25}}{24} = \frac{576}{25 \times 24} = \frac{576}{600}$

Let DE = y.
Let EC = z.
Then $\frac{DE}{EC} = \frac{y}{z} = \frac{576}{600}$

$z = \frac{600}{576}y$
$h = 6\frac{18}{25} = y + z$
$\frac{168}{25} = y + \frac{600}{576}y$
$\frac{168}{25} = \frac{576}{576}y + \frac{600}{576}y = \frac{1176}{576}y$
$y = \frac{168 \times 576}{1176 \times 25} = \frac{21 \times 576}{147 \times 25} = \frac{3 \times 576}{21 \times 25} = \frac{576}{7 \times 25} = \frac{576}{175}$ **<u>Ans.</u>**

<u>Team Round</u>

1. Prime numbers 2, 3, 5 and 7 are used to replace a, b, c, and d in the multiplication table. The four products are found and then added together. Find the greatest possible value of the sum.

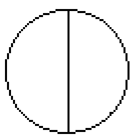
×	*a*	*b*
c		
d		

The products of combinations of the 4 numbers, two at a time are:
2 × 3 = 6
2 × 5 = 10
2 × 7 = 14
3 × 5 = 15
3 × 7 = 21
5 × 7 = 35
Clearly, we would want to get rid of the lowest two, but we can’t. If a = 2, then we must have ac and ad as products. So we can place 2 and 3 together as a and b and not have to deal with ab. But that means c and d would be 5 and 7 and we would then lose cd or 35. If we put 2 and 7 together and 3 and 5 together we’ll at least remove two of the lower products. Let’s try that.

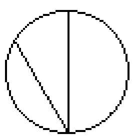
×	2	7
3	6	21
5	10	35

The sum of the products are:
6 + 21 + 10 + 35 = 72 **Ans.**

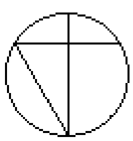
2. A pizza is cut 5 times before being removed from the pan. What is the maximum number of pieces that can be made which contain none of the pizza's outer crust?
The first cut won't create any pieces that don't have crust.



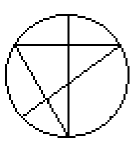
Neither will the second cut.



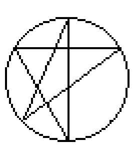
The third cut can create 1 piece without crust.



The fourth cut can add two more pieces.



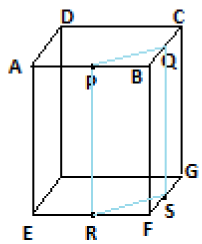
And finally, the fifth piece can add 3 more pieces (each successive cut adds one more than the previous cut did).



1 + 2 + 3 = 6 **Ans.**

3. There are 3 die, one red, one green and one white. Patrick tosses each and creates equations, that may or may not be correct , of the form:
red die value – green die value = white die value
Patrick rolls and creates 4 − 3 = 2
If you rotate each die a quarter turn, the equation can be made correct. How many correct equations are possible?
When 4 is on top of the red die, turning the die one turn gives you access to: {1, 2, 5, 6}.
When 3 is on top of the green die, turning the die one turn gives you access to {1, 2, 5, 6}.
When 2 is on top of the white die, turning the die one turn gives you access to {1, 3, 4, 6}.
(Uh, no, I wasn't going to turn that last set white…)
The first value, for the red die, can't be a 1 since we can't have a result that's 0 or less. So let's start with 2.
2 − 1 = 1
Now let's use 5 for the red die.
5 − 1 = 4
5 − 2 = 3
Finally, let's use 6 for the red die.
6 − 2 = 4
6 − 5 = 1
That's a total of 5. **Ans.**

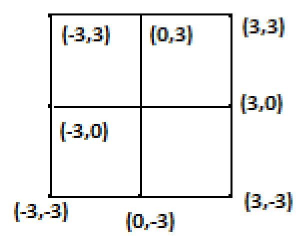
4. A square prism has dimensions 5' × 5' × 10', where ABCD is a square. AP = ER = 2 and QC = SG = 1. The plane containing PQ and RS slices the original prism into two new prisms. We must find the volume of the larger of these two prisms.



The volume of the entire square prism is
5 × 5 × 10 = 250
If AP = 2, then PB = 5 − 2 = 3
If QC = 1, then BQ = 5 − 1 = 4
The area of triangle PBQ is
$\frac{1}{2} \times 3 \times 4 = 6$
The volume of the triangular prism is
6 × 10 = 60
Therefore, the volume of the other prism is 250 − 60 = 190 **Ans.**

5. What is the sum of all real numbers x such that
$4^x - 6 \times 2^x + 8 = 0$
$2^{2x} - 6 \times 2^x + 8 = 0$
Let $y = 2^x$
Then, we can rewrite the equation as:
$y^2 - 6y + 8 = 0$
$(y - 4) \times (y - 2) = 0$
y = 4, y = 2
Substituting 2^x for y, we have:
$2^x = 4$
x = 2
and
$2^x = 2$
x = 1
1 + 3 = 3 **Ans.**

6. What is the area of the region bounded by the graph of
$|x - y| + |x + y| = 6?$
Since we're dealing with absolute values we're looking at the sums at the edges of 0 + 6 and 6 + 0.
Let's start with
$|x - y| = 0$ and $|x + y| = 6$
Clearly, x = y from the first relationship and $x = y = 3$ or $x = y = -3$ from the second.
Now let's do:
$|x - y| = 6$ and $|x + y| = 0$
$x = -y$ from the second relationship and $x = 3, y = -3$ or $x = -3, y = 3$ from the first.
Now let's look at
$|x - y| = 3$ and $|x + y| = 3$
The values for x and y that satisfy these relationships are
$x = 3, y = 0$
$x = -3, y = 0$
$x = 0, y = 3$
$x = 0, y = -3$
When we start plotting the values we end up with a square with side 6.



Any other values of x and y that satisfy the relationships will be on the outside of this square. The area of the region bounded by the relationships is
$6 \times 6 = 36$ **Ans.**

7. How many collections of 6 positive odd integers have a sum of 18?
Let's start with as many 1's as possible:
5 ones
1 + 1 + 1 + 1 + 1 + 13 = 18
4 ones
1 + 1 + 1 + 1 + 3 + 11 = 18
1 + 1 + 1 + 1 + 5 + 9 = 18
1 + 1 + 1 + 1 + 7 + 7 = 18
3 ones
1 + 1 + 1 + 3 + 3 + 9 = 18
1 + 1 + 1 + 3 + 5 + 7 = 18
1 + 1 + 1 + 5 + 5 + 5 = 18

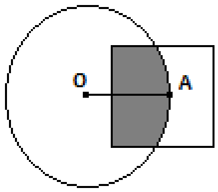
2 ones
1 + 1 + 3 + 3 + 3 + 7 = 18
1 + 1 + 3 + 3 + 5 + 5 = 18
1 one
1 + 3 + 3 + 3 + 3 + 5 = 18
Finally we can do all 6 integers as 3:
3 + 3 + 3 + 3 + 3 + 3 = 18
Any higher number and we won't be able to have six odd integers to sum to 18. We are done.
1 + 3 + 3 + 2 + 1 + 1 = 11 **Ans.**

8. In how many ways can 15,015 be represented as the sum of two or more consecutive positive integers written in ascending order?
First, let's determine the maximum number of consecutive integers we can have. The equation for determining the largest number of consecutive integers is to start from 1 or:
$\frac{n \times (n+1)}{2} = 15015$
$n^2 + n - 30030 = 0$
If m is the solution to this equation we would have:
$(n+m)(n+m-1) = 30030$
The value for m will be pretty close to the square root of 30030 or approximately 173.
Therefore, we can't have more than 173 consecutive numbers to sum to 15015.
Now let's look at odd numbers of consecutive positive integers.
If there are 3 integers we could call them x − 1, x and x + 1. Then
x − 1 + x + x + 1 = 15015
3x = 15015
And if there were 5 integers we could call them x − 2, x − 1, x, x + 1 and x + 2.
Then x − 2 + x − 1 + x + x + 1 + x + 2 = 5x = 15015
You can now see the pattern. When n is an odd integer, we get
*n*x = 15015
x can only have an integral value when *n* is a factor of 15015.
Let's factor 15015.
15015 = 3 × 5005 =
3 × 5 × 1001 =
3 × 5 × 7 × 143 =
3 × 5 × 7 × 11 × 13
Remember that any factor of 15015 no greater than 173 would work.
One single factor:
3, 5, 7, 11, 13
Combinations of two factors:
15, 21, 33, 39, 35, 55, 65, 77, 91, 143
Combinations of three factors:
105, 165 only
Combinations of four factors:
None – every combination is bigger than 173. So far we have 5 + 10 + 2 = 17 sets of consecutive integers.
Now, let's look at even values.
If n = 2, then
x + x + 1 = 15015
2x + 1 = 15015
2x = 15014
x = 7507 #18
If n = 4, then let's try and get as many additions to cancel out:
x − 1 + x + x + 1 + x + 2 = 15015
4x + 2 = 15015
4x = 15013
This won't work and no multiple of 4 will work. (When n = 2, we had 2x +1; and when n = 4, we had 4x + 2 and when n = 3 we will have 6x + 3 – we must have an odd number to subtract from 15015 to make an even number).
If n = 6, then 6x + 3 = 15015
6x = 15012
x=2502 (That's #19 and 6 = 3 × 2)
n = 10:
10x + 5 = 15015
10x = 15010
x = 1501 (That's #20 and 10 = 5 × 2)

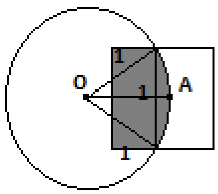
14x + 7 = 15015
14x = 15008
x = 1072 (That's #21 and 14 = 7 × 2)
18x + 9 = 15015
18x = 15006
x = 833.6667 – This one didn't work.
18 = 3 × 3 × 2
22x + 11 = 15015
22x = 15004
x = 682 (That's #22 and 22 = 11 × 2)
26x + 13 = 15015
26x = 15002
x = 577 (That's #23 and 26 = 13 × 2)
30x + 15 = 15015
30x = 15000
x = 500 (That's #24 and 30 = 5 × 3 × 2)
34x + 17 = 15015
34x = 14998
x = 441.117
Okay. There's a pattern here and that is that the even factors are composed of 2 and factors of 15015. We've already identified 3, 5, 7, 11, 13 and 15. So the rest of the possible factors are:
42 (21 × 2) : #25
66 (33 × 2) : #26
78 (39 × 2) : #27
70 (35 × 2) : #28
110 (55 × 2) : #29
130 (65 × 2): #30
154 (77 × 2): #31
Any other values multiplied by 2 will be greater than 173.
31 **<u>Ans.</u>**

9. A positive integer is *squarish* if it contains the digits of the squares of its digits in order but not necessarily contiguous. What is the smallest squarish number that includes at least one digit greater than 1?
Let's look at the first squares.
1, 4, 9, 16, 25, 36, 49, 64, 81, 100
1, 2, 3, 4, 5, 6, 7, 8, 9, 10
Let's start with 12. $1^2 = 1$ (that is why they said includes at least one digit greater than 1).
12 is next and $2^2 = 4$
124 is next and $4^2 = 16$
1246 is next and $6^2 = 36$
12436 is next and we can switch this to 12346 without violating the requirements. Now $3^2 = 9$
123469 is next and $9^2 = 81$
8123469 isn't next because $8^2 = 64$ and the 4 is before the 6. Let's move that.
8123649 – this works but can we make the value smaller by moving some of the numbers around?
2364981 – I just moved 81 to the end. That's good but can I move anything again?
Yes, it won't hurt if I switch the 9 and 81.
2364819
13 can be reduced in a similar way to 381649 – that is definitely less than 2364981.
14 can be reduced to 348169.
15 can be reduced to 2364819
16 is can be reduced to 381649.
17 can be reduced to 8136479 – no matter what else we do this number is greater than 381649.
18 is next and leads to 381649
19 leads to 381649
And what you can see is that the choice becomes which two numbers leads to the smallest number of integers that we need to represent the squares.
381649 **<u>Ans.</u>**

10. A square of side length 1 inch is drawn with its center A on a circle O of radius 1 inch so that a side of the square is perpendicular to OA. What is the area of the shaded region?



We can create an equilateral triangle by drawing two radii from O to the points of the circle that intersect with the square.



The height of the triangle is $\frac{\sqrt{3}}{2}$ and the portion of that height that is in the grey colored area in the triangle is $\frac{\sqrt{3}}{2}-\frac{1}{2}$.

This makes the area of the rectangular portion of the grey area

$$\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)\times 1=\frac{\sqrt{3}}{2}-\frac{1}{2}=0.3660254$$

Now for the area of the chord:

The triangle that was drawn is an equilateral triangle making the angle at O to be 60°. Therefore, the area of the sector is $\frac{1}{6}\pi r^2=\frac{\pi}{6}$

The area of the equilateral triangle is

$$\frac{1}{2}\times\frac{\sqrt{3}}{2}\times 1=\frac{\sqrt{3}}{4}$$

Thus, the area of the chord is:

$$\frac{\pi}{6}-\frac{\sqrt{3}}{4}\approx 0.5235983-0.433012701\approx 0.0905855$$

Adding the area of the rectangular portion of the grey area to the chord gives:

$0.3660254+0.0905855\approx$
$0.45661099\ \approx 0.46$ **Ans.**