MATHCOUNTS®

2014 State Competition Solutions

Are you wondering how we could have possibly thought that a Mathlete® would be able to answer a particular Sprint Round problem without a calculator?

Are you wondering how we could have possibly thought that a Mathlete would be able to answer a particular Target Round problem in less 3 minutes?

Are you wondering how we could have possibly thought that a Mathlete would be able to answer a particular Team Round problem with less that 10 sheets of scratch paper?

The following pages provide solutions to the Sprint, Target and Team Rounds of the 2014 MATHCOUNTS® State Competition. Though these solutions provide creative and concise ways of solving the problems from the competition, there are certainly numerous other solutions that also lead to the correct answer, and may even be more creative or more concise! We encourage you to find numerous solutions and representations for these MATHCOUNTS problems.

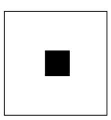
Special thanks to volunteer author Mady Bauer for sharing these solutions with us and the rest of the MATHCOUNTS community!

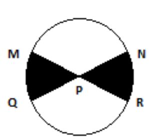
2014 State Competition

Sprint Round

1. A mouse weighs 25 grams and a dog weighs 5000 grams. The weight of the dog is $\frac{5000}{25} = 200$ times the weight of the mouse. 200 **Ans.**

2. $|x - 3| = 4$
First the positive value:
$x - 3 = 4$
$x = 7$
Now the negative value:
$-(x - 3) = 4$
$-x + 3 = 4$
$x = -1$
The two solutions are 7 and -1.
The product of the solutions is:
$7 \times -1 = -7$ **Ans.**

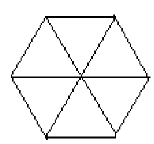
3. A square target has a side length of 16. The smaller square has a side length of 4. Find the probability that the dart lands in the smaller square.

The area of the smaller square is:
$4 \times 4 = 16$
The area of the larger square is:
$16 \times 16 = 256$
The probability that the dart lands in the smaller square is:
$\frac{16}{256} = \frac{1}{16}$ **Ans.**

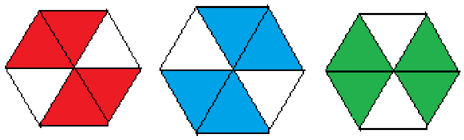
4. The diameter of Circle P is 2. The measure of ∠MPN is 120°. The area of the shaded region is *k*π. Find k.


Given that the measure of ∠MPN is 120°, then that is also the measure of ∠QPR.
120 + 120 = 240
360 – 240 = 120
The measure of ∠MPQ and ∠NPR are both the same, i.e. 60°.
Each section denoted by those angles has an area of 1/6 the area of the circle for a total of 1/3 the area of the circle.
Given that the diameter of the circle is 2, then the radius is 1.
The area of the circle is $\pi r^2 = \pi$. 1/3 the area of the circle is (1/3)π. Therefore, *k* = 1/3 **Ans.**

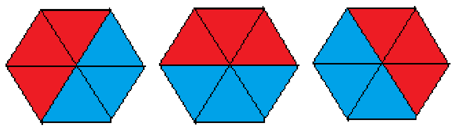
5. $f(x) = \frac{f(x+1)}{2}$
$f(2) = 4$
Find the value of $f(10) - f(7)$
From the first equation:
$f(x + 1) = 2f(x)$
$f(3) = 2f(2) = 8$
$f(4) = 2f(3) = 16$
$f(5) = 2f(4) = 32$
Anyone see what the value of f(x) really is?
$f(x) = 2^x$
Therefore, $f(7) = 2^7 = 128$ and $f(10) = 2^{10} = 1024$
$f(10) - f(7) = 1024 - 128 = 896$ **Ans.**

6. If n is the number of seconds in a day, find the largest prime factor of n.
The number of seconds in the day is:
24 × 60 × 60 =
(2 × 2 × 2 × 3) × (2 × 2 × 3 × 5) × (2 × 2 × 3 × 5)
Clearly, 5 is the largest prime.
5 **Ans.**

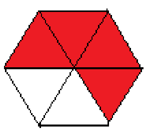
7. How many convex polygons with an even number of sides are in the figure shown?

A convex polygon has all angles less than 180°. A polygon must have at least 3 sides but we are only dealing with 4 sides so let's start with polygons of size 4. The first 4-sided shape we can look at is a rhombus. There are 6 of those.



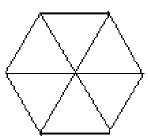
Next we have the trapezoid. There are 6 of those:



Now 6 sided polygons. We have some "irregular" shaped polygons, like this one..



But…remember that we are looking for convex polygons. It's obvious that we have an angle that is larger than 180°. This doesn't qualify.
So what is left, is the original hexagon..



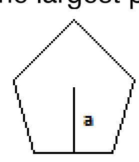
6 + 6 + 1 = 13 **Ans.**

8. Stage 1 has the number 1. Obviously the sum is 1.
Stage 2 has the numbers 2, 3 and 1, 2. The sum is 8.
Stage 3 has the numbers, 3, 4, 5 and 2, 3, 4 and 1, 2, 3. The sum is 12 + 9 + 6 = 27.
Stage 4 has the numbers 4, 5, 6, 7 and 3, 4, 5, 6 and 2, 3, 4, 5 and 1, 2, 3, 4 for a sum of 22 + 18 + 14 + 10 = 64.
See the pattern? The sum of the numbers in a stage? If the stage is n, then the sum of the numbers is n^3.
The sum of the numbers for stage 7 is $7^3 = 343$. **Ans.**

9. The sequence 1, 3, 4, 7, 11, 18, 29 is constructed by adding the two previous terms (after the first two terms). How many of the first 100 terms of this sequence are multiples of 5?
So far, none. Let's go a little further but we only need to put down the one's column.
7, 6, 3, 9, 2, 1, 3, 4, 7, 1, 8 – wait a minute. We're into repeating the ones column. We never got the value 0 or 5. Therefore, there will be 0 terms in this sequence that are multiples of 5.
0 **Ans.**

10. 3 different gifts were bought for 3 children. The gifts were wrapped after their names were put on them. Find the probability that no child opens a gift labeled with his or her name.
Name the gifts 1, 2, and 3.
Name the kids A, B and C.
A was meant to have 1, B was meant to have 2 and C was meant to have 3.
If A gets present 2, then B can get 3 and C can get 1.
If A gets present 3, then B can get 1 and C can get 2. That's 2 choices. And those same choices apply if we start with B or C. I.e.,
A→2, B→3, C→1 and
A→3, B→1, C→2.
There are 3! = 6 combinations.
2/6 = 1/3 **Ans.**

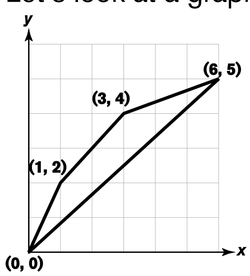
11. The sides of 3 similar regular pentagons are in the ratio of 2:5:7. The area of the smallest pentagon is 40. Find the area of the largest pentagon.



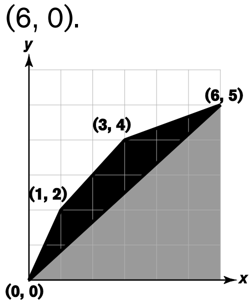
The formula for the area of a regular pentagon is $\frac{5}{2}sa$ where s is the side of the pentagon and a is the apothem (a line drawn from the center of the pentagon to the midpoint of a side).
Let $2x$ = the side of the smallest pentagon. Then the apothem for the pentagon is:

$40 = \frac{5}{2}sa = \frac{5}{2} \times 2xa = 5xa$
$8 = xa$
$a = \frac{8}{x}$
Since the ratio of the smallest to the middle pentagon is 2:5, the side of the middle pentagon is 5x and the apothem of the middle pentagon is:
$\frac{8}{x} \times \frac{5}{2} = \frac{40}{2x} = \frac{20}{x}$
Finally, the ratio of the middle to the largest pentagon is 5:7.
Therefore, the side of the largest is 7x.
The apothem of the largest pentagon is:
$\frac{20}{x} \times \frac{7}{5} = \frac{28}{x}$
The area of the largest pentagon is:
$\frac{5}{2} \times 7x \times \frac{28}{x} = \frac{5}{2} \times 7 \times 28 = 5 \times 7 \times 14 =$
490 **Ans.**

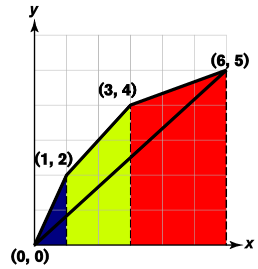
12. Find the area of the quadrilateral with vertices (0,0), (1,2), (3,4) and (6,5).
Let's look at a graph of the quadrilateral.



It's going to be tough to try and find the area of the quadrilateral directly, so let's consider the pentagon with vertices (0,0), (1,2), (3,4), (6,5) and (6, 0). As shown, the area of the quadrilateral is the area of the pentagon minus the area of the right triangle with vertices (0, 0), (6, 5) and (6, 0).



To determine the area of the pentagon, let's partition it into two trapezoids and a right triangle, as shown.



Therefore, the area of the quadrilateral is
[(1/2) × 1 × 2] + [((2 + 4)/2) × 2] +
[((4 + 5)/2) × 2] – [(1/2) × 6 × 5] =
1 + 6 + 27/2 – 15 = (14 + 27 – 30)/2 =
11/2 = 5 1/2 **Ans.**

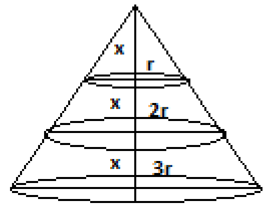
13. The integers *a*, *b*, *c*, *d* and *e* form an arithmetic sequence. Their sum is 440. Find the largest possible value of e.
A sequence is an arithmetic sequence if the terms differ by the same value.
Let *x* = that difference. Then
$b = a + x$
$c = a + 2x$
$d = a + 3x$
$e = a + 4x$
$a + b + c + d + e =$
$a + a + x + a + 2x + a + 3x + a + 4x =$
$5a + 10x = 440$
$a + 2x = 88$
To find the largest value of e, a must be as small as possible. The numbers in the series are non-negative. Suppose $a = 0$.
$0 + 2x = 88$
$2x = 88$
$x = 44$
$e = a + 4x = 0 + (4 \times 44) = 176$ **Ans.**

14. "789XYZ" is a 6-digit integer consisting of six distinct digits. It is divisible by 7, 8 and 9. Find the three-digit integer "XYZ".
A number that is divisible by 9 must have the sum of the digits add up to a value that is divisible by 9.
7 + 8 + 9 = 24
X + Y + Z must be divisible by 3 but not divisible by 9, otherwise the entire number will not be divisble by 9.
We have combinations of
{1, 2, 3} (sums to 6)
{1, 2, 9}, {1, 3, 8}, {1, 4, 7}, {1, 5, 6},

{2, 3, 7}, {2, 4, 6}, {3, 4, 5} (sums to 12)
{3, 4, 8}, {3, 5, 7}, {4, 5, 6} (sums to 15)
{5, 7, 9}, {6, 7, 8} (sums to 21)
{7, 8, 9} (sums to 24)
But we can't use any groups that have a 7, 8 or 9 in them because all numbers are distinct. That takes us down to:
{1, 2, 3}
{1, 5, 6}
{2, 4, 6}
{3, 4, 5}
{4, 5, 6}
A number is divisible by 8 if its last three digits are divisble by 8. Obviously, the last digit must also be an even number. So we can look at each of these combinations choose an even number to be the ones digit and see if the three digit number is divisible by 8.
{1, 2, 3} gives us 132 and 312. **312** is divisible by 8.
{1, 5, 6} gives us 156 and 516. Neither are divisible by 8.
{2, 4, 6} gives us 246, 264, 426, 462, 624 and 642. **264** and **624** are divisible by 8.
{3, 4, 5} gives us 354 and 534. Neither are divisible by 8.
{4, 5, 6} gives us 456, 546, 564 and 654. **564** is divisble by 8.
So we are left at looking at each of
789,312
789,264
789,624
789,564
Only 789,264 is divisble by 7 as well.
264 **Ans.**

15. The roots of the equation $x^2 + 6x + k = 0$ are in the ratio of 2:1. Find k.
If we looked at $x^2 + 6x + 9 = 0$, the two roots are x = 3. No ratio of 2:1 there.
But if we looked at
$(x + 2)(x + 4) = x^2 + 6x + 8 = 0$
the roots are -2 and -4 which do satisfy the requirement that the roots are in the ratio of 2:1. Therefore, k = 8. **Ans.**

16. A cone is sliced by planes parallel to its base into 3 pieces of equal height. Find what fraction of the original volume is in the middle piece.



Let h = the height of the entire cone.
Let x = the length of the height between the top of the cone and the first plane, between the first and second planes and between the second and third planes.
Then $h = 3x$
If r = the radius of the smallest cone, then $2r$ and $3r$ are the radii of the other two cones since the sides of the cone and the planes all form similar triangles.
The volume of a cone is $\frac{1}{3}\pi r^2 h$.
The volume of the entire cone is
$\frac{1}{3}\pi \times (3r)^2 \times 3x = 9\pi r^2 x$
The volume of the smallest cone is
$\frac{1}{3}\pi \times (r)^2 \times x = \frac{1}{3}\pi r^2 x$
The volume of the middle cone is
$\frac{1}{3}\pi \times (2r)^2 \times 2h = \frac{8}{3}\pi r^2 x$
The volume of the middle piece is the volume of the middle cone minus the volume of the smallest cone.
$\frac{8}{3}\pi r^2 - \frac{1}{3}\pi r^2 x = \frac{7}{3}\pi r^2 x$
The fraction of the original volume is
$\frac{\frac{7}{3}\pi r^2 x}{9\pi r^2 x} = \frac{\frac{7}{3}}{9} = \frac{7}{27}$
$\frac{7}{27}$ **Ans.**

17. A six-sided die has two red, two blue and two yellow faces. The die is rolled 3 times. Find the probability of getting each color once.
The probability of getting a particular color when the die is rolled is
$\frac{2}{6} = \frac{1}{3}$

The probability of getting a certain color the first time (say red), a different color (say blue) the second time and a different color again (say yellow) the third time is:
$\frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$
There are 3! = 6 different combinations of red, blue and yellow. Therefore, the probability of getting each color once is:
$\frac{6}{27} = \frac{2}{9}$ **Ans.**

18. A code has to include 4 distinct digits in the range of 1 through 5. The positive difference between any two consecutive digits must be at least 2. Find how many codes can be used.
Start with 1.
1 3 5 2
1 4 2 5
1 5 2 4 – That's 3 possibilities.
Now start with 2.
2 4 1 3
2 4 1 5
2 5 1 3
2 5 1 4
2 5 3 1 – That's 5 possibilities.
Now start with 3:
3 1 4 2
3 1 5 2
3 5 1 4
3 5 2 4 – That's 4 possibilities.
Now start with 4:
4 1 3 5
4 1 5 2
4 1 5 3
4 2 5 1
4 2 5 3 – That's 5 possibilities.
And finally, start with 5
5 1 4 2
5 2 4 1
5 3 1 4 – That's 3 possibilities.
That's a total of 3 + 5 + 4 + 5 + 3 = 20 combinations.
20 **Ans.**

19. The composition, to the nearest whole percent of a class is 43% male and 57% female. What is the minimum number of students that could be in the class?
Nearest whole percent is the operative term here.
Clearly, if we have 2 students, one male and one female, it's 50% for either.
3 students would be $\frac{1}{3}$ or $\frac{2}{3}$. That won't work.
4 students obviously won't work either nor would 5 students or 6 students.
What about 7 students?
$\frac{3}{7} \approx 0,.4285 \ldots \approx 43\%$
7 **Ans.**

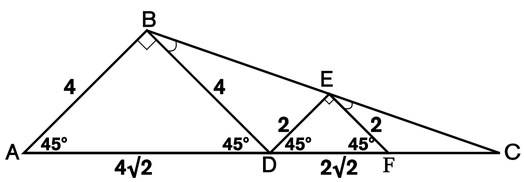
20. Express $27_{10} \times 314_5$ in base 5.
I don't mind adding in a base different from 10 but multiplication is nasty. Let's convert 314_5 into base 10, perform the multiplication and then turn it back into a value in base 5.
$314_5 = (3 \times 5^2) + (1 \times 5^1) + (4 \times 5^0) =$
$75 + 5 + 4 = 84$
$84 \times 27 = 2268$
$2268 = 1875 + 375 + 15 + 3 =$
$(625 \times 3) + (125 \times 3) + (5 \times 3) +$
$(1 \times 3) =$
$(3 \times 5^4) + (3 \times 5^3) + (3 \times 5^1) +$
$(3 \times 5^0) = 33033_5$ **Ans.**

21. $x^2 + \frac{1}{x^2} = 3, x > 0$
Find the value of $x + \frac{1}{x}$
$\left(x + \frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2}$
$\left(x + \frac{1}{x}\right)^2 = 3 + 2 = 5$
$x + \frac{1}{x} = \sqrt{5}$ **Ans.**

22. A bag contains red marbles and blue marbles. The probability that two marbles chosen are red is 1/5. The probability that two marbles chosen are blue is 1/5. Determine the number of marbles in the bag.
Let *x* = the number of red marbles.
x is also the number of blue marbles.
$\frac{x}{2x} \times \frac{x-1}{2x-1} = \frac{1}{5}$
$\frac{x^2 - x}{4x^2 - 2x} = \frac{1}{5}$

$5x^2 - 5x = 4x^2 - 2x$
$x^2 - 3x = 0$
$x(x-3) = 0$
x cannot be 0 since we have to have "some" marbles here.
$x = 3$
2x is the number of marbles or
6 **Ans.**

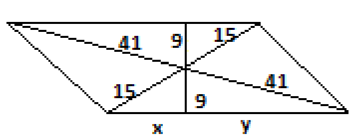
23. Triangles ABD and DEF are isosceles right triangles. A, D, F and C are collinear and B, E and C are collinear.
AB = BD = 4. ED = EF =2. Find the length of AC.



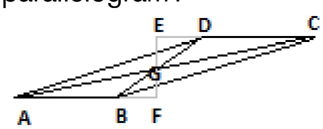
Since ∠DBE is congruent to ∠FEC, triangles ABC and DEC are similar (AA). Therefore, AB/DE = AC/DC → 4/2 = $(4\sqrt{2} + 2\sqrt{2}AD + FC)/(2\sqrt{2} + FC)$ → $8\sqrt{2} + 4 \times FC = 8\sqrt{2} + 4\sqrt{2} + 2 \times FC$ → $2 \times FC = 4\sqrt{2}$ → $FC = 2\sqrt{2}$ units. Thus, $AC = 4\sqrt{2} + 2\sqrt{2} + 2\sqrt{2} = 8\sqrt{2}$. **Ans.**

24. How many digits are in the integer representation of 2^{30}?
Looking at the powers of 2, we have
2 4 8 16 32 64 128 256 512 1024 2048 4096 8192 16384 32768
The number of digits in each can be written as:
1 1 1 2 2 2 3 3 3 etc.
Therefore, the number of digits needed to represent 2^{30} is 10. **Ans.**

25. The diagonals of a parallelogram are 82 and 30. One altitude is 18. Find the smallest possible area of the parallelogram.



The area of a parallelogram is bh where b is the base of the parallelogram and h is the altitude. To find the area, we need to find the base, *b*. Break that into two pieces, *x* and *y*, which are formed by dropping the altitude. The intersection of the diagonals also bisects the altitude.
$b = x + y$
$y^2 + 9^2 = 41^2$
$y + 81 = 1681$
$y^2 = 1681 - 81 = 1600$
$y = 40$
$x^2 + 9^2 = 15^2$
$x^2 + 81 = 225$
$x^2 = 225 - 81 = 144$
$x = 12$
$b = x + y = 12 + 40 = 52$
The area of the parallelogram is:
$bh = 52 \times 18 = 936$
But does this give us the smallest possible area? Probably not since this is the obvious solution.
In the diagram above, the altitude was completely contains within the parallelogram. Is there a way in which the altitude is not contained within the parallelogram?



This happens when the two of the parallel lines of the parallelogram are entirely divorced from each either – i.e, move DC down and move AB up and no point of AB will touch a point of DC.
h = EF and EG = GF = 9
DB = 30 and DG = GB = 15
AC = 82 and AG = GC = 41
Let x = AF. Then
$x^2 + 9^2 = 41^2$
$x^2 + 81 = 1681$
$x^2 = 1681 - 1600$
$x = 40$
Let y = BF. Then
$y^2 + 9^2 = 15^2$
$y^2 + 81 = 225$
$y^2 = 225 - 81 = 144$
$y = 12$
AF = AB + BF
$x = AB + y$

Now, instead of adding *x* and *y*, we get to subtract *y* from *x*.
$40 = AB + 12$
$AB = 40 - 12 = 28$
AB is the base so the area is
$28 \times 18 = 504$
504 is definitely smaller than 936.
504 **Ans.**

26. 7 prizes are to be distributed between 3 people with a guarantee that each person gets at least one prize. Find the number of ways the prizes can be distributed.
Prizes can be given to the three people in the following combinations:
1, 1, 5
1, 2, 4
1, 3, 3
2, 2, 3

For 1, 1, 5 there are $\frac{7!}{5!2!} = \frac{7\times6}{2} = 21$ ways for the 5 gifts to be given to one particular person. To make this more understandable, assume person #3 has the 5 gifts. Then person #1 gets gift A and person #2 gets gift B (A and B are not part of the 5) or person #1 gets gift B and person #2 gets gift A. So, for one set of 5 gifts going to person #3, there are two combinations for persons #1 and #2. Therefore, for 21 combinations of the 5 gifts there are $21 \times 2 = 42$. This is just for Person #3 receiving the 5 gifts. There are 42 more for Person #1 and Person #2 receiving the 5 gifts, respectively.
$42 \times 3 = 126$
For 1, 2, 4 there are $\frac{7!}{4!3!} = \frac{7\times6\times5}{3\times2\times1} = 35$ ways to choose the 4. Again, assume that Person #3 receives the 4 gifts. Assume that Person #1 receives the single gift and Person #2 receives the two gifts. There are $\frac{3!}{2!1!}$=3 ways to allocate the 3 gifts between Person #1 and Person #2. If we now let Person #1 receive two gifts and Person #2 receive the single gift, that's another 3 combinations for a total of 6 combinations while Person #3 receive the 4 gifts. $35 \times 6 = 210$ combinations when Person #3 receives the 4 gifts. Adding in combinations when Person #1 receives the 4 gifts and Person #2 receives the 4 gifts we get a total of $210 \times 3 = 630$ combinations.
For 1, 3, 3, let's have Person #3 receive the single gift. There are $\frac{6!}{3!3!} = \frac{6\times5\times4}{3!} = 20$ ways to allocate the first group of 3 prizes, say, to Person #1. But allocation of 3 prizes to Person #1 also allocates the other 3 prizes to Person #2. This allocation covers all combinations for both Person #1 and Person #2 (e.g., Person #3 gets present 1, Person #1 gets presents 2-4 and Person #2 gets 5-7 or Person #1 gets presents 5-7 and Person #2 gets 2-4 etc.) Therefore, we have 20 combinations per single present for Person #3 for a total of 7×20 combinations when Person #3 gets the single present. This ends up being a total of $140 \times 3 = 420$ combinations when Person #1 and person #2 get the single present.
Finally, for 2, 2, 3, there are 35 ways to allocate the group of 3 and $\frac{4!}{2!2!} = \frac{4\times3}{2} = 6$ ways to allocate the sets of 2 presents to the other two people.
$35 \times 6 = 210$ ways and $210 \times 3 = 630$ total combinations.
The final total of combinations is 126 + 630 + 420 + 630 = 1806 ways **Ans.**

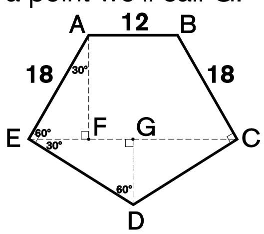
27. The fourth degree polynomial equation of $x^4 - 7x^3 + 4x^2 + 7x - 4 = 0$ has 4 roots, a, b, c and d. Find the value of the sum
$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$
$x^4 - 7x^3 + 4x^2 + 7x - 4$ is of the form:
$a_4x^4 + a_3x^3 + a_2x^2 + a_1x^1 + a_0$ where $a_4 = 1, a_3 = -7, a_2 = 4, a_1 = 7$ and $a_0 = -4$
The quickest way to solve this is to understand what the product of $(x-a) \times (x-b) \times (x-c) \times (x-d)$ looks like.

$(x-a)\times(x-b)\times(x-c)\times(x-d)=$
$x^4-x^3(a+b+c+d)+$
$x^2(ab+ac+ad+bc+bd+cd)-$
$x(abc+abd+acd+bcd)+abcd$
$a_4=1$
$a_3=-7=a+b+c+d$
Note that this is the sum of the roots.
$a_2=4=ab+ac+ad+bc+bd+cd$
Note that this is the sum of the product of two of the roots.
$a_1=7=abc+abd+acd+bcd$
Note that this is the sum of the product of three of the roots.
Finally, $a_0=-4=abcd$
This is the product of all 4 roots.
Now let's look at
$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}$
$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=\frac{b+a}{ab}+\frac{d+c}{cd}=$
$\frac{bcd+acd+abd+abc}{abcd}=$
$\frac{abc+abd+acd+bcd}{abcd}=\frac{a_1}{a_0}=\frac{7}{4}$ **Ans.**

28. Find the remainder when 11^{12} is divided by 13. One way to solve this problem without a calculator is by using modular arithmetic.
Since 11 mod 13 = −2 mod 13, we know that 11^{12} mod 13 = $(-2)^{12}$ mod 13. But $(-2)^{12}=2^{12}$, so we just need to find the remainder when 2^{12} is divided by 13.
2^{12} = 4096, and 13 × 315 = 4095. So, the remainder is 1 **Ans.**

29. $f(n)=n^2+1$ if n is odd
$f(n)=\frac{n}{2}$ if n is even
For how many integers from 1 to 100 does $f\left(f(\ldots f(n))\right)=$ 1 for some number of applications of f?
Let's start with 1.
$f(1)=1^2+1=1+1=2$
$f(f(1))=f(2)=\frac{2}{2}=1$
At this point, we're going to repeat 2, 1, 2, 1, etc.
Now start with 2.
$f(2)=\frac{2}{2}=1$
$f(f(2)=f(1)=2$
And again, we'll just repeat 1, 2.
Now start with 3.
$f(3)=3^2+1=9+1=10$
$f(f3))=f(10)=\frac{10}{2}=5$
$f\left(f(f(3))\right)=f(5)=5^2+1=26$
$f\left(f\left(f(f(3))\right)\right)=f(26)=\frac{26}{2}=13$
Continued application of the function f will only result in larger and larger values. Therefore, 3 is not a part of this (and none of the rest of the odd values).
Now start with 4.
$f(4)=\frac{4}{2}=2$
$f(f(4))=f(2)=\frac{2}{2}=1$
So 4 works. We already know that 5 will not so let's look at 6.
$f(6)=\frac{6}{2}=3$
$f(f(6))=f(3)=10$
We now have enough information. For n=1, 2, 4, we can have the situation where multiple applications of f will result in the value 1. Odd values starting with 3 will never result in the value of the function being 1 because the value just gets bigger and bigger. That leaves multiples of 4.
$f(8)=\frac{8}{2}=4$
We already know that f(4) works so 8 is also good.
$f(12)=\frac{12}{2}=6$
And now we can see that any multiple of 3 (and, of course, 4) cannot be used. Basically, what we are down to are powers of 2. 1, 2, 4, 8, 16, 32, 64
7 **Ans.**

30. In pentagon ABCDE, ∠E and ∠C are right angles and $m\angle D = 120$, AB=12, AE=BC=18 and FD=DC. Find ED.
Let's start by constructing segment EC. Then we'll draw a perpendicular segment from A that intersects EC at a point we'll call F. Lastly, we'll draw another perpendicular from D that intersects EC at a point we'll call G.



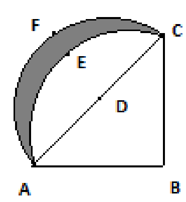
The figure shows that when these additional segments drawn, 30-60-90 right triangles AFE and DGE are created. For triangle AFE, since AE = 18, using the properties of 30-60-90 right triangles, we conclude that EF = 9. Notice that FG = AB/2 = 12/2 = 6, which means EG = 9 + 6 = 15. For 30-60-90 triangles the ratio of the hypotenuse to the longer leg is 2/√3. Therefore, for triangle DGE, the hypotenuse, side ED, and the longer leg, side EG, must be in the ratio 2/√3.
We have ED/EG = 2/√3 → ED/15 = 2/√3 → ED = 15 × 2/√3 = 30/√3.
Rationalizing the denominator, we get 30/√3 × √3/√3 = 30√3/3 = 10√3. **Ans.**

Target Round

1. A birthday present costs $94.94. Sales tax is 5.5%. Find the cost of the present before the tax was applied.
Let x = the cost of the present before tax is applied.
Then $94.94 is 1.055 times the cost of the present or 1.055x.
$94.94 = 1.055x$
$x = \frac{94.94}{1.055}$
$x = \$88.99$ **Ans.**

2. The sides of the scales balance. Each shape has a weight from 1 to 10 pounds. All 7 shapes total 32 pounds. Find the weight of the rectangle given that the weight of the rhombus minus two pounds equals the weight of the rectangle plus the weight of the circle.
Let c = the weight of the circle.
Let t = the weight of the triangle.
Let r = the weight of the rectangle.
Let h = the weight of the rhombus.
$3c + h + r + 2t = 32$
$h = r + c + 2$
Since the scale balances:
$r + c$ balances $2t$
$r + c = 2t$
$2c + h$ balances $r + c + 2t$
$2c + h = r + c + 2t =$
$r + c + r + c = 2r + 2c$
$2c + h = 2r + 2c$
$h = 2r = r + c + 2$
$r = c + 2$
$c = r - 2$
Going back to the equation representing the sum of all the weights:
$3c + 2r + r + r + c = 32$
$4c + 4r = 32$
$c + r = 8$
$r - 2 + r = 8$
$2r - 2 = 8$
$2r = 10$
$r = 5$ **Ans.**

3. Arc AFC is drawn on isosceles right triangle ABC. The arc is drawn with center D which is the midpoint of AC. The length of AB is 90 feet. Find the area of the shaded region.



Given that AB is 90, so is BC since the triangle is an isosceles right triangle. Therefore, the length of AC is
$\sqrt{90^2 + 90^2} = 90\sqrt{2}$
D is the midpoint of AC.

AD = DC = $45\sqrt{2}$ which is the radius of the circle at point D.
Now let's find some areas.
The area of the quarter-circle with center B and arc AEC is:
$\frac{\pi r^2}{4} = \frac{\pi \times 90^2}{4} = \frac{8100\pi}{4} = 2025\pi$
The area of triangle ABC is
$\frac{1}{2} \times 90 \times 90 = 45 \times 90 = 4050$
Finally, the area of the half circle with center D and arc AFC is
$\frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left(45\sqrt{2}\right)^2 = \frac{4050\pi}{2} = 2025\pi$
The area of the shaded region is:
2025π – (2050π – 4050) = 4050 **Ans.**

4. The sum of seven positive integers is 42. The mean, median and mode are consecutive integers. Find the largest possible integer in the list.
That the sum of the seven integers is 42 tells us that $\frac{42}{7} = 6$ is the mean.
Start by assuming that the mean is the smallest value. Then we can have the median at 7 and the mode at 8 or the mode at 7 and the median at 8.
If the mode is 8 and the median is 7, then 1, 2, 3, 7, 8, 8, x would be the situation with the largest x for that scenario.
1 + 2 + 3 + 7 + 8 + 8 = 29
29 + x = 42
x = 13

In the next scenario with the mode at 7 we would have:
1 + 7 + 7 + 8 + 9 + 10 + x
1 + 7 + 7 + 8 + 9 + 10 = 42
We're at an impasse – this can't work.
So far, the largest value is 13.
Now, assume that the mean (6) is in the middle. Then the mode could be 5 and the median 7 or the median could be 7 and the mode 5.
Starting with a mode of 5, we would have:
1, 5, 5, 7, 8, 9, x
1 + 5 + 5 + 7 + 8 + 9 = 35 + x
x = 7 and that breaks the requirements.
If the median is 5, then we have
1, 2, 3, 5, 7, 7, x
1 + 2 + 3 + 5 + 7 + 7 = 25
25 + x = 42
x = 17
So far, this is the best value.
Finally the last scenario is that the mean is the largest value. Then we have a mode of 4 and median of 5 or a mode of 5 and median of 4.
Starting with the mode of 4,
1, 4, 4, 5, 6, 7, x
1 + 4 + 4 + 5 + 6 + 7 = 27
27 + x = 42
x = 15
That's smaller than 17.
Finally we have:
1, 2, 3, 4, 5, 5, x
1 + 2 + 3 + 4 + 5 + 5 = 20
20 + x = 42
42 – 20 = 22
And that's the best choice!
22 **Ans.**

5. A grid contains all of the integers 1 through 100 in ten rows of ten numbers. 1 through 10 are on the top row, 11 through 20 are in the second row, etc.
A token placed on the board can be moved to the right, up, left, down, up and right, up and left, down and left or down and right. The token is placed on 27.
The first move made is down 1 and to the right 1.
Down 1 moves the token to 37 and to the right 1 moves it to 38.
The second move is up 1. That moves the token up to 28.
The third move is left 1. That moves the token to 27.
The fourth move is right 1. That moves the token to 28.
The fifth move is up one and to the left 1.
Up 1 moves the token to 18. Left one moves the token to 17.
The sixth move is down 1 and left 1.
Down 1 moves the token to 27. Left1 moves the token to 26.
The seventh move is down 1 and left 1.
Down 1 moves the token to 36. Left 1

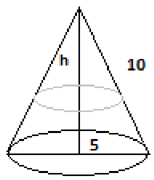
moves the token to 35.
The eighth move is right 1.
Right 1 moves the token to 36.
The ninth move is up 1 and left 1.
Up 1 moves the token to 26. Left 1 moves the token to 25.
The tenth move is down 1 and left 1.
Down 1 is 35. Left 1 is 34.
The eleventh move is down 1.
This moves the token to 44.
The twelfth move is up 1 and right 1.
Up 1 moves the token to 34.
Right 1 moves the token to 35.
The thirteenth move is down 1 left 1.
Down 1 moves the token to 45.
Left 1 moves the token to 44.
The fourteenth move is down 1 right 1.
Down 1 moves the token to 54.
Right 1 moves the token to 55.
The fifteenth move is up 1.
Up 1 moves the token to 45.
45 **Ans.**

6. Mandy shoots free throws with the probability of $p < 1/2$. The probability of her shooting six free throws and making exactly half of them is $4^4/5^5$. Find the value of p.
The probability of making a free throw is p.
The probability of missing the free throw is $1 - p$.
There are $\frac{6!}{3!3!} = \frac{6\times5\times4}{3\times2\times1} = 20$ combinations where she makes exactly half of the six free throws. The probability of making 3 free throws out of six is:
$p^3 \times (1-p)^3 \times 20$
Now that just looks ugly. Let's go at this a different way – in particular, backwards.
We know that $\frac{4^4}{5^5}$ is the probability of making 3 out of 6 free throws. That's actually all 20 scenarios. So the probability of making 3 out of 6 free throws using a specific scenario (like getting the first 3 free throws and missing the last 3) is
$$\frac{\frac{4^4}{5^5}}{20} = \frac{4^4}{5^5\times4\times5} = \frac{4^3}{5^6}$$
$$\frac{4^3}{5^6} = \frac{1}{5}\times\frac{1}{5}\times\frac{1}{5}\times\frac{4}{5}\times\frac{4}{5}\times\frac{4}{5}$$
So either p is 1/5 or p is 4/5. But we are told that $p < 1/2$, so p must be 1/5 **Ans.**

7. An ant takes four steps each 1 unit in length from the origin. Each step is either forward or backward, right or left. Find the probability that the ant's fourth step places the ant back at the origin.
Let's list the ways the ant can get back to the origin.
First by returning to the origin after 2 steps.
- right, left, right, left
- left, right, left, right
- forward, backward, forward, backward
- backward, forward, backward, forward
- right, left, up, down
- right, left, down, up
- right, left, left, right
- left, right, up, down
- left, right, down, up
- left, right, right, left
and similarly for forward and backward.
That's 16 movements.
The ant can also go around a one unit square.
- right, backward, left, forward
- right, forward, left, backward
- forward, right, backward, left
- forward, left, backward, right
and similarly for left and backward.
That's a total of 8 ways. Add that to the 16 we listed before and that's 24 ways.
The ant can go halfway around the square and then come back.
- right, backward, forward, left
- right, forward, backward, left
etc. for another 8 ways for a total of 32 ways.
The ant can go 2 steps in one direction and then come back the same way, e.g.,
- left, left, right, right etc. for a total of 36 more.
32 + 4 = 36
In moving each step, the ant has a $\frac{1}{4}$ probability that he moves in a certain direction. This means that choosing 4

specific movements is
$\frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{256}$
$\frac{36}{256} = \frac{9}{64}$ **Ans.**

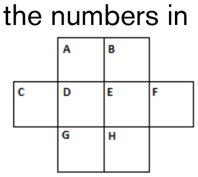
8. The cone has a base radius of 5 and a slant height of 10. The top cone will be removed so that the volume of the remaining frustum is $\frac{1}{3}$ of the original cone. Find the total surface area of the frustum.



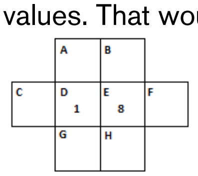
First find the volume of the full cone. This means that we need to know the height of the cone.
$r = 5$
$h^2 + r^2 = h^2 + 5^2 = 10^2$
$h^2 + 25 = 100$
$h^2 = 75$
$h = \sqrt{75} = 5\sqrt{3}$
Let h_1 = the height of the top cone.
Let h_2 = the height of the frustum.
$h = h_1 + h_2$
Let s = the slant height of the whole cone.
Let s_1 = the slant height of the top cone.
Let s_2 = the slant height of the frustum.
$s = s_1 + s_2$
Let r_1= the radius of the top cone.
The volume of the entire cone is
$$\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(5)^2 \times 5\sqrt{3} = \frac{125}{3}\sqrt{3}\pi$$
If the volume of the frustum is 1/3 the volume of the entire cone, the volume of the smaller cone (the tope part that is removed) is 2/3 of the entire volume of the cone. The triangle formed by the sides *h*, *r* and *s* is similar to the triangle formed by side h_1, r_1 and s_1. The volume of the top cone is
$\frac{1}{3}\pi(r_1)^2 h_1 = \frac{2}{3}\left(\frac{1}{3}\pi r^2 h\right) = \frac{2}{9}\pi r^2 h$
$\pi(r_1)^2 h_1 = \frac{2}{3}\pi r^2 h$
$(r_1)^2 h_1 = \frac{2}{3} r^2 h$
Since the triangles are similar the smaller triangle is proportionally smaller than the larger triangle.
Let x = the proportion.
$h_1 = xh$
$r_1 = xr$
$(r_1)^2 h_1 = (xr)^2 xh = x^3 r^2 h = \frac{2}{3} r^2 h$
$x^3 = \frac{2}{3}$
$x = \sqrt[3]{\frac{2}{3}} \approx 0.87358$
$s_2 = s - s_1 = s - xs = (1 - x)s$
$s_2 = (1 - 0.87358)s = (0.12642) \times 10$
$s_2 = 1.2642$
$h_2 = h - h_1 = h - hx = (1 - x)s$
$h_2 = (0.12642) \times 5\sqrt{3} \approx 1.094829$
$r_1 = xr = 0.87358 \times 5 \approx 4.3679$
The surface area of a frustum is
$\pi[s_2(r + r_1) + r^2 + {r_1}^2] =$
$\pi[1.2642(5 + 4.3679) + 5^2 + (4.3679)^2]$
$= \pi[1.2642(9.3679) + 25 + 19.07855] =$
$\pi[11.84289 + 44.07855] =$
$55.92145\pi \approx 175.6822 \approx 176$

Team Round

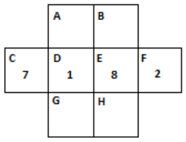
1. The numerals 1 to 8 are arranged so that no two consecutive integers touch at a side or on a corner. Find the product of the numbers in boxes C and F.



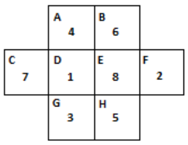
Boxes D and E have the most adjacent boxes. Therefore, they should have the numbers that have the least adjacent values. That would be 1 and 8.



Boxes C and F should be as far away from the values in Boxes D and E as possible. That would be 7 and 2.



And that would leave 4 and 6 for Boxes A and B and 3 and 5 for Boxes G and H.



We could also switch the values in A and B with those in G and H. In any event the values in boxes C and F are $7 \times 2 = 14$ **Ans.**

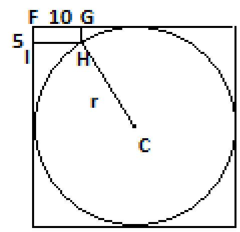
2. The chart was generated using the equation
$y = ax^2 + bx + c$
Find the value of a.

x	y
-5	8
-3	5
-1	4
1	5
3	8

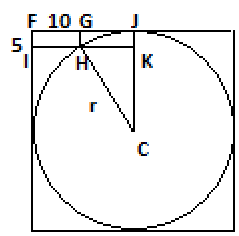
Let's start by using the first set of values of x and y in the equation.
$8 = a(-5)^2 + (b \times -5) + c$
$8 = 25a - 5b + c$ (Eq. 1)
Now do the same with the second set of values.
$5 = a(-3)^2 + (b \times -3) + c$
$5 = 9a - 3b + c$ (Eq. 2)
Now do the same with the third set of values.
$4 = a(-1)^2 + (b \times -1) + c$
$4 = a - b + c$ (Eq. 3)
Subtract the second equation from the first:
$3 = 16a - 2b$ (Eq. 4)
Subtract the third equation from the second.
$1 = 8a - 2b$ (Eq. 5)
Subtract the fifth equation from the fourth equation.
$2 = 8a$
$a = \frac{2}{8} = \frac{1}{4}$ **Ans.**

3. $f(x) = x + 2$
$g(x) = x^2$
For what value of x does
$f(g(x)) = g(f(x))$?
$f(g(x)) = f(x^2) = x^2 + 2$
$g(f(x) = g(x+2) = (x+2)^2$
$x^2 + 2 = (x+2)^2 = x^2 + 4x + 4$
$2 = 4x + 4$
$4x = -2$
$x = -\frac{2}{4} = -\frac{1}{2}$ **Ans.**

4. The side lengths of the small rectangle are 5 and 10. The lower right vertex of the rectangle is on the circle. Find the radius of the circle.



Let's start by naming the center of the circle, C. The rectangle is FGHI and H is the point where the rectangle touches the circle. Then CH = r, the radius.
FI = GH = 5
FG = IH = 10.



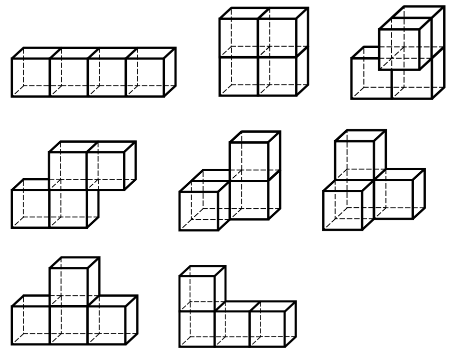
Draw another radius of point C to the midpoint of the top of the square, at point J. Extend rectangle FGHI to make the larger rectangle FJKI. GJKH is also a rectangle.
JK = GH = FI = 5
HK = IK – IH
IK is also the radius so
HK = r – 10
KC = JC – JK = r – 5
$r^2 = (r-10)^2 + (r-5)^2$
$r^2 = r^2 - 20r + 100 + r^2 - 10r + 25$
$r^2 = 2r^2 - 30r + 125$
$r^2 - 30r + 125 = 0$

$(r-25)(r-5)=0$
$r=25$ or $r=5$
r can't be 5 since FI = 5 and FI < JC.
25 **Ans.**

5. How many positive 3-digit integers are palindromes and multiples of 11?
Let's look at the multiples of 11 between 100 and 200.
110, 121, 132, 143, 154, 165, 176, 187, 198
Only 121. Any other 3-digit integers that are multiples of 11 (and palindromes) will be such that if *abc* is the 3-digit integer, *a* = *c*, and *a* + *c* = *b*
That would be 242, 363, 484.
Once we get to 500 that won't work.
Now the requirement is that
a = *c* and *b* = *a* + *c* − 11
5 + 5 = 10 so we can't subtract 11 from it. But continuing on:
616, 737, 858, 979
1 + 3 + 4 = 8 **Ans.**

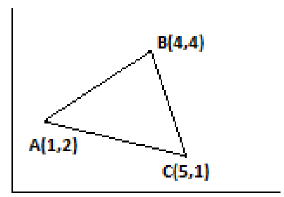
6. When traffic is heavy you can average 30 mi/hour to arrive on time. If it is light, you can average 55 mi/hour and arrive 1 hour and 15 minutes early. Find the distance to the football stadium.
Let's say that it takes x hours at 30 miles an hour to get to the stadium. That's a total of 30x miles. At 55 mi/hour it takes 1 hour and 15 minutes less than x hours or
$x-1\frac{1}{4}$
$30x=55\left(x-\frac{5}{4}\right)$
$30x=55x-\frac{5\times 55}{4}$
$25x=55x-\frac{5\times 55}{4}$
$25x=\frac{5\times 55}{4}$
$x=\frac{5\times 55}{25\times 4}=\frac{11}{4}=2\frac{3}{4}$ hr
Therefore the distance is:
$30\times 2.75=82.5$ **Ans.**

7. Four 1 cm cubes are joined face to face in all ways to form geometric solids. How many solids are possible?

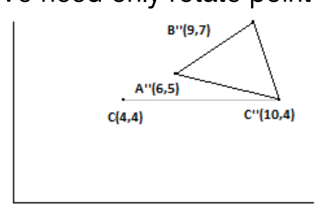


As shown, 8 geometric solids can be formed. 8 **Ans.**

8. Triangle ABC has vertices A(1,2), B(4,4) and C(5,1). It is translated horizontally to the right 5 units, then vertically up 3 units. Finally the triangle is rotated 90° around (4, 4). Find the coordinates of image C.



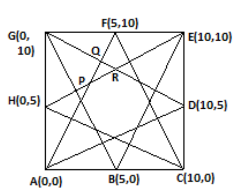
Translate 5 units to the right. This means adding 5 to the *x*-coordinate. We have three new points:
A'(6,2), B'(9,4), C'(10,1)
Now translate 3 units up. This means adding 3 to the y-coordinate. We have three new points:
A''(6,5), B''(9.7), C''(10,4)
We are asked to rotate the triangle A''B''C'' 90° clockwise around (4,4) and find the coordinates of the new point C. We need only rotate point C.



To do the rotation, note that C'' is directory to the right of C. If you imagine that C is the center of a clock then C'' can be considered as 3:00. We need to find the coordinates of 6:00 (since we're moving the pointer clockwise). Point C'' is 6 units to the right of point C. To make the new point "seem" like 6:00 it must be on a line 6 units directly below point C. That point is (4, −2). **Ans.**

9. *a* and *b* are positive integers.
$\frac{1}{2}+\frac{1}{a}=\frac{1}{3}+\frac{1}{b}$
Find the sum of all possible values of a.
$\frac{1}{b}-\frac{1}{a}=\frac{1}{2}-\frac{1}{3}=\frac{3}{6}-\frac{2}{6}=\frac{1}{6}$
Clearly, *b* = 2 and *a* = 3 is one solution.
Let's look at *b.*
$1-\frac{1}{a}=\frac{1}{6}\rightarrow\frac{6}{6}-\frac{1}{6/5}=\frac{1}{6}$
Thus, *a* = 6/5 which is not an integer.
Now let's look at *b* = 3.
$\frac{1}{3}-\frac{1}{a}=\frac{1}{6}\rightarrow\frac{1}{3}-\frac{1}{6}=\frac{1}{6}$
So *a* can also be 6. So far, the sum is
3 + 6 = 9
Now let's look at *b* = 4.
$\frac{1}{4}-\frac{1}{a}=\frac{1}{6}\rightarrow\frac{6}{24}-\frac{2}{24}=\frac{4}{24}\rightarrow\frac{1}{4}-\frac{1}{12}=\frac{1}{6}$
So *a* can be 12. The sum is now
9 + 12 = 21.
Now let's look at *b* = 5.
$\frac{1}{5}-\frac{1}{a}=\frac{1}{6}\rightarrow\frac{6}{30}-\frac{1}{30}=\frac{1}{6}\rightarrow\frac{1}{5}-\frac{1}{30}=\frac{1}{6}$
So *a* can be 30. The sum is now
21 + 30 = 51.
Now let's look at *b* = 6.
Oops, that's $\frac{1}{6}$,
We are done. 51. **Ans.**

10. In square ABCD, each vertex is connected to the midpoints of its two opposite sides. Find $\frac{QR}{PQ}$.



Let *s* = the side of the square. Then if we were to graph the square, it would have the coordinates as displayed in the diagram – if you used *s* = 10.
First, let's find the equation for the line APQF.
$y = mx + b$ using point A
$0 = (m \times 0) + b$
$b = 0$
Using point F:
$10 = 5m$
$m = \frac{10}{5} = 2$
Therefore, the equation for line APQF is
$y = 2x$
Now the equation for line DRQG:
Using point D:
$5 = 10m + b$
Using point G:
$10 = 0m + b$

$b = 10$
$5 = 10m + 10$
$10m = -5$
$m = -\frac{5}{10} = -\frac{1}{2}$
The equation for DRQG:
$y = -\frac{1}{2}x + 10$
Finally, the equation for the line HPRE:
Using point H:
$5 = (m \times 0) + b$
$b = 5$
Using point E:
$10 = (m \times 10) + 5$
$5 = 10m$
$m = \frac{5}{10} = \frac{1}{2}$
The equation for line HPRE is
$y = \frac{1}{2}x + 5$
We can find the coordinates of P if we find the intersection of lines HPRE and APQF.
$y = \frac{1}{2}x + 5$
$y = 2x$
$2x = \frac{1}{2}x + 5$
$2x - \frac{1}{2}x = 5$
$\frac{3}{2}x = 5$

$x = \frac{10}{3}$

$y = 2x = 2 \times \frac{10}{3} = \frac{20}{3}$

So, point P is $\left(\frac{10}{3}, \frac{20}{3}\right)$

Now for point Q. That is at the intersection of lines APQF and DRQG.

$y = -\frac{1}{2}x + 10$

$y = 2x$

$2x = -\frac{1}{2}x + 10$

$2x + \frac{1}{2}x = 10$

$\frac{5}{2}x = 10$

$x = \frac{20}{5} = 4$

$y = 2x = 2 \times 4 = 8$

The coordinates of point Q are $(4{,}8)$

Finally, point R is at the intersection of lines HPRE and DRQG.

$y = -\frac{1}{2}x + 10$

$y = \frac{1}{2}x + 5$

$-\frac{1}{2}x + 10 = \frac{1}{2}x + 5$

$x = 5$

$y = \left(\frac{1}{2} \times 5\right) + 5 = \frac{15}{2}$

The coordinates of point R are $\left(5, \frac{15}{2}\right)$.

Now that we have the coordinates, we can get the lengths of PQ and QR.

$PQ = \sqrt{\left(4 - \frac{10}{3}\right)^2 + \left(8 - \frac{20}{3}\right)^2} =$

$\sqrt{\left(\frac{12-10}{3}\right)^2 + \left(\frac{24-20}{3}\right)^2} =$

$\sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{4}{3}\right)^2} =$

$\sqrt{\frac{4}{9} + \frac{16}{9}} = \sqrt{\frac{20}{9}} = \frac{1}{3}\sqrt{20} = \frac{2}{3}\sqrt{5}$

$QR = \sqrt{(5-4)^2 + \left(\frac{15}{2} - 8\right)^2} =$

$\sqrt{(1)^2 + \left(\frac{15-16}{2}\right)^2} =$

$\sqrt{1 + \left(-\frac{1}{2}\right)^2} + \sqrt{1 + \frac{1}{4}} = \sqrt{\frac{5}{4}} = \frac{1}{2}\sqrt{5}$

Finally:

$\frac{QR}{PQ} = \frac{\frac{1}{2}\sqrt{5}}{\frac{2}{3}\sqrt{5}} = \frac{\frac{1}{2}}{\frac{2}{3}} = \frac{1}{2} \times \frac{3}{2} = \frac{3}{4}$ **Ans.**