MATHCOUNTS®

2015 State Competition Solutions

Are you wondering how we could have possibly thought that a Mathlete® would be able to answer a particular Sprint Round problem without a calculator?

Are you wondering how we could have possibly thought that a Mathlete would be able to answer a particular Target Round problem in less 3 minutes?

Are you wondering how we could have possibly thought that a Mathlete would be able to answer a particular Team Round problem with less that 10 sheets of scratch paper?

The following pages provide solutions to the Sprint, Target and Team Rounds of the 2015 MATHCOUNTS® State Competition. Though these solutions provide creative and concise ways of solving the problems from the competition, there are certainly numerous other solutions that also lead to the correct answer, and may even be more creative or more concise! We encourage you to find numerous solutions and representations for these MATHCOUNTS problems.

*Special thanks to volunteer author **Mady Bauer** for sharing these solutions with us and the rest of the MATHCOUNTS community!*

2015 State Competition

Sprint Round

1. Given: A bike costs $240. Jared has $60 and saves $9 per week.
   Find: The number of weeks it will take Jared to save the rest of the money for the bike.
   Subtracting 240 – 60 = 180, gives us the amount he needs to save. At $9 per week, the number of weeks it will take is
   $\frac{180}{9} = 20$ **Ans.**

2. Given: $2 = \frac{a}{20} = \frac{60}{b}$
   Find: $a + b$
   $2 = \frac{a}{20}$
   $a = 40$
   $2 = \frac{60}{b}$
   $2b = 60$
   $b = 30$
   So $a + b = 40 + 30 = 70$ **Ans.**

3. Given: $f(x) = 2x^2 - 3x + 7$ with domain {−2, −1, 3, 4}
   Find: The largest integer in the range of *f*.
   Substitute each of the domain values into the function to determine the range values.
   $f(-2) = 2(-2)^2 - 3(-2) + 7 = 8 - (-6) + 7 = 21$
   $f(-1) = 2(-1)^2 - 3(-1) + 7 = 2 - (-3) + 7 = 12$
   $f(3) = 2(3)^2 - 3(3) + 7 = 18 - 9 + 7 = 16$
   $f(4) = 2(4)^2 - 3(4) + 7 = 32 - 12 + 7 = 27$
   The largest value is 27. **Ans.**

4. Given: First two terms are 10 and 20. The rest of the terms are the average of the preceding terms.
   Find: The 2015th term
   Let's see if we can generalize.
   3rd term: $\frac{10+20}{2} = \frac{30}{2} = 15$
   4th term: $\frac{10+20+15}{3} = \frac{45}{3} = 15$
   5th term: $\frac{10+20+15+15}{4} = \frac{60}{4} = 15$
   We can now generalize. The average of the first two terms was 15. And after that, we continue to add 15 because the average is always 15. Therefore, the value of the 2015th term also will be 15. **Ans.**

5. Given: a graph showing the water temperature *T* at time *t* for a pot of water heated to 100°.
   Find: The average of how many degrees the temperature increased each minute during the first 8 minutes.
   At time $t = 0$, the temperature is $T = 20$.
   At time $t = 8$, the temperature is $T = 100$. So the average increase in temperature per minute was $\frac{100-20}{8} = \frac{80}{8} = 10$ **Ans.**

6. Given the following table:

x	*y*
1	1
s	4
t	*t* + 4

   and $y = \frac{3x-1}{2}$
   Find: *t*
   Using the last line of the table:
   $t + 4 = \frac{3t-1}{2}$
   $2t + 8 = 3t - 1$
   $t = 9$ **Ans.**

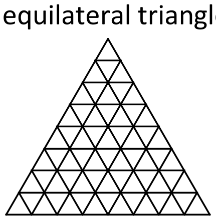
7. Given: First plan has a $24 registration fee, a monthly rate of $5 and a charge of $0.75 for each download.
   Second plan has a $15 registration fee, a fixed monthly rate of $3 and a charge of $1.50 for each download.
   Find: The number of downloads for which the two plans cost the same for a year.
   Let *x* be the number of downloads for which the two plans cost the same.

We have

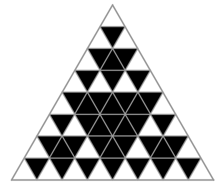
$24 + (12 \times 5) + 0.75x =$
$15 + (12 \times 3) + 1.50x$
$84 + 0.75x = 51 + 1.50x$
$0.75x = 33$
$x = 44$ **Ans.**

8. Given: Length of a rectangle is 3 times the width. A second rectangle decreases the length by 9 and increases the width by 4. Both rectangles have the same area.
   Find: The perimeter of the second rectangle.
   Let l = length of the first rectangle.
   Let w = width of the first rectangle.
   Then $l = 3w$
   The length of the second rectangle is $3w - 9$, and its width is $w + 4$
   Now, since both areas are the same we have:
   $(3w)(w) = (3w - 9)(w + 4)$
   $3w^2 = 3w^2 - 9w + 12w - 36$
   $3w = 36$
   $w = 12$
   The second rectangle has length$3w - 9 = 36 - 9 = 27$ and width $w + 4 = 12 + 4 = 16$.
   Its perimeter, then, is $2(27 + 16) = 2(43) = 86$ **Ans.**

9. Given: Equilateral triangle ABC has downward pointing black triangles whose vertices are at the midpoints of the sides of a larger upward pointing white triangle.
   Find: The fraction of the area of triangle ABC that is white.
   First, break up the triangle into 64 smaller congruent equilateral triangles as shown.



If we place this figure of equilateral triangles over triangle ABC, as shown below, we can count the white triangles to determine what fraction of triangle ABC they represent.



Now since there are 27 white triangles, the fraction of triangle ABC that they represent is $\frac{27}{64}$. **Ans.**

10. Given: an isosceles triangle with sides $5x + 20$, $3x + 76$, $x + 196$
    Find: The greatest possible perimeter.
    Since the triangle is isosceles, two sides must be equal in length. If $5x + 20 = 3x + 76$, then $2x = 56$ and $x = 28$. If $3x + 76 = x + 196$, then $2x = 120$ and $x = 60$. Finally, if $5x + 20 = x + 196$, then $4x = 176$ and $x = 44$. The greatest value for x is 60, so the greatest possible perimeter occurs with side lengths $5x + 20 = 5(60) + 20 = 320$, $3x + 76 = 3(60) + 76 = 256$ and $x + 196 = 60 + 196 = 256$. That perimeter is $320 + 256 + 256 = 832$ **Ans.**

11. Given: An arithmetic sequence with $a_3 = 165$ and $a_{12} = 615$
    Find: n if $a_n = 2015$
    There is a common difference between terms in an arithmetic sequence. The difference between a_3 and a_{12} is $615 - 165 = 450$. The common difference between each of the nine pairs of terms from a_3 to a_{12} is $450 \div 9 = 50$. Since $2015 - 615 = 1400$ and $\frac{1400}{50} = 28$, to get from $a_{12} = 615$ to $a_n = 2015$ this common difference is applied 28 times. So $n = 12 + 28 = 40$ **Ans.**

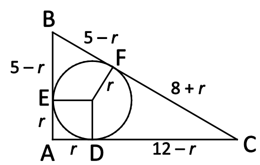
12. Given: $\sqrt{x+7} = 2 + \sqrt{x}$
Find: The value of *x*.
$x + 7 = \left(2 + \sqrt{x}\right)^2 = 4 + 4\sqrt{x} + x$
$7 = 4 + 4\sqrt{x}$
$3 = 4\sqrt{x}$
$9 = 16x$
$x = \frac{9}{16}$ **Ans.**

13. Given: The line perpendicular to 2*x* – 2*y* = 2 with the same *y*-intercept is graphed.
Find: The sum of its *x*- and *y*-intercept
Rewriting 2*x* – 2*y* = 2 in the form *y* = *mx* + *b*, we get *y* = *x* – 1 and see that the slope is *m* = 1 and the *y*-intercept is *b* = −1.
The line perpendicular to *y* = *x* – 1 has a slope of $-\frac{1}{m} = -1$. The *y*-intercept is the same so the equation of the perpendicular line is *y* = −*x* – 1. The *x*-intercept, which is found when *y* = 0, is *x* = −1. The sum of the *x*- and *y*-intercepts of the perpendicular line is: −1 + (−1) = −2 **Ans.**

14. Find: The units digit of the sum of the squares of the integers from 1 to 2015.
Let's look at the squares of the first 10 numbers to see if we can generalize.
1, 4, 9, 16, 25, 36, 49, 64, 81, 100
And the next 10 numbers, we have
121, 144, 169, 196, 225, 256, 289, 324, 361, 400.
The one's digit repeats in every set of 10 squares.
The sum of the one's digits in each set of 10 squares is:
1 + 4 + 9 + 6 + 5 + 6 + 9 + 4 + 1 + 0 = 45
Now, let's look at the sums of the squares of integers from 1 to *n* where 1 < *n* < 10.
1, 5, 14, 30, 55, 91, 140, 204, 285, 385
The fifth term is 55. If we add the next 10 squares to get to the 15th term, we add 45 (actually we only care about the 5) and the 15th term will end in 0 (i.e., 5 + 5). Similarly, the 25th term will end in 5. And so on. The one's digit of the terms that end in 5 will cycle back and forth from 5 to 0 to 5 etc. Therefore, the one's digit of the 2015th term is 0 (the 2005th term would have a one's digit of 5). 0 **Ans.**

15. Given: Amber takes one or two vitamins a day. She has a total of 10 vitamins.
Find: How many different ways Amber can take the vitamins.
Start with 2 a day.
$2 + 2 + 2 + 2 + 2$ - 1 way
Now let her take 2 vitamins a day for 4 days and 1 vitamin a day for 2 days, i.e.,
$2 + 2 + 2 + 2 + 1 + 1$
There are $\frac{6!}{4!2!} = \frac{6\times5}{2} = 15$ ways to do this.
Now take 2 vitamins a day for 3 days and 1 vitamin a day for 4 days, i.e.,
$2 + 2 + 2 + 1 + 1 + 1 + 1$
$\frac{7!}{4!3!} = \frac{7\times6\times5}{3\times2\times1} = 35$ ways
Now take 2 vitamins a day for 2 days and 1 vitamin a day for 6 days, i.e.,
$2 + 2 + 1 + 1 + 1 + 1 + 1 + 1$
$\frac{8!}{6!2!} = \frac{8\times7}{2\times1} = 28$ ways.
Now take 2 vitamins a day for 1 day and 1 vitamin a day for 8 days, i.e.,
$2 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$
$\frac{9!}{8!1!} = 9$ ways
Finally, there is 1 way to take one vitamin a day for 10 days. The total number of ways to take the 10 vitamins as prescribed is
1 + 9 + 28 + 35 + 15 + 1 = 89 **Ans.**

16. Given: a circle inscribed in a triangle with sides of length 5, 12 and 13.
Find: The radius of the circle.
The triangle is a 5-12-13 right triangle.



Draw a radius perpendicular to each side of the triangle. The point of intersection between each of these radii and the triangle also is the point of tangency between the circle and that side of the triangle.
Let r be the radius of the circle.
Side AB of length 5 is divided into two segments AE = r and EB = 5 – r.
Side AC of length 12 is divided into two segments of AD = r and DC = 12 – r.
Two segments tangent to a circle that intersect at a point outside the circle are equal in length, so EB = BF.
Hypotenuse BC of length 13 is divided into two segments of lengths BF = 5 − r and FC = 13 – (5 – r) = 8 + r.
Segments DC and FC are both tangent to the circle and intersect at C, so DC = FC and 12 – r = 8 + r

$2r = 4$

$r = 2$ **Ans.**

17. Given: A bag has blue, red and purple gumballs in the ratio of 2:3:4. 5 red balls are added to the bag making the probability of drawing a red gumball 40%.
Find: The number of gumballs now in the bag.
Let x = the original number of gumballs in the bag.
The number of blue gumballs is $\frac{2}{9}x$.
The number of red gumballs is $\frac{3}{9}x$.
The number of purple gumballs is $\frac{4}{9}x$.
Now, add the 5 red gumballs.

$$\frac{\frac{3}{9}x + 5}{x + 5} = 40\% = \frac{4}{10} = \frac{2}{5}$$

$$\frac{15}{9}x + 25 = 2x + 10 = \frac{18}{9}x + 10$$

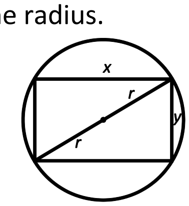
$$\frac{3}{9}x = 25 - 10 = 15$$

$$\frac{1}{3}x = 15$$

$x = 45$; This is the original number of gumballs. The number of gumballs now in the bag is x + 5 = 45 + 5 = 50. **Ans.**

18. Given: $x^2 + px + 2p = 0$ with solutions: $x = a$ and $x = b$
$bx^2 + cx + d = 0$ with solutions: $x = a + 2$ and $x = b + 2$
Find: d
For the first quadratic equation we have
$(x - a)(x - b) = 0$
$x^2 - x(a + b) + ab$
Therefore, $p = -(a + b)$
$2p = ab$
$-2(a + b) = ab$
$2(a + b) + ab = 0$
For the second equation we have
$(x - (a + 2))(x - (b + 2)) = 0$
$x^2 - x(a + b + 4) + 2(a + b) + ab + 4 = 0$
So $c = -(a + b + 4)$ and $d = 2(a + b) + ab + 4$.
From the previous equation, we know that $2(a + b) + ab = 0$, so $d = 4$. **Ans.**

19. Given: The perimeter of a rectangle is 22. The rectangle is inscribed in a circle of with area 16π.
Find: The area of the rectangle.
Let x = the length of the rectangle.
Let y = the width of the rectangle.
Let r = the radius.



We are asked to find the area of the rectangle, $A = xy$.

The circle has area 16π so

$\pi r^2 = 16\pi$

$r^2 = 16$

$r = 4$

By the Pythagorean Theorem

$x^2 + y^2 = 8^2$

$x^2 + y^2 = 64$

Since the perimeter of the rectangle is 22,

$2(x + y) = 22$

$x + y = 11$

Squaring both sides, we get

$(x + y)^2 = 11^2$

$x^2 + 2xy + y^2 = 121$

Substituting 64 for $x^2 + y^2$ gives us

$64 + 2xy = 121$

$2xy = 57$

$xy = 28.5$ **Ans.**

20. Given: The units and tens digits of a two-digit number are the tens and units digit of another two-digit number. The product of the two integers is 4930.

    Find: The sum of the two integers.

    4930 = 493 × 2 × 5 = 17 × 29 × 2 × 5

    By inspection, the two numbers must be 85 = 17 × 5 and 58 = 29 × 2. And 85 + 58 = 143 **Ans.**

21. Given: 8 blue socks, 6 red socks, 4 black socks and 2 orange socks.

    Find: The probability of selecting 2 socks of the same color.

    There are a total of 8 + 6 + 4 + 2 = 20 socks.

    The probability of selecting two orange socks:

    $\frac{2}{20} \times \frac{1}{19} = \frac{2}{380} = \frac{1}{190}$

    The probability of selecting two black socks:

    $\frac{4}{20} \times \frac{3}{19} = \frac{12}{380} = \frac{6}{190}$

    The probability of selecting two red socks:

    $\frac{6}{20} \times \frac{5}{19} = \frac{30}{380} = \frac{15}{190}$

    The probability of selecting two blue socks:

    $\frac{8}{20} \times \frac{7}{19} = \frac{56}{380} = \frac{28}{190}$

    The total probability of selecting two socks of the same color is:

    $\frac{1}{190} + \frac{6}{190} + \frac{15}{190} + \frac{28}{190} = \frac{50}{190} = \frac{5}{19}$ **Ans.**

22. Given: Large bars weigh 8 g. Medium bars weigh 6 kg. Small bars weigh 3 kg. Iron, nickel and lead are present in the ratio 4:1:3 in each large bar, 2:1:3 in each medium bar and 1:1:1 in each small bar. A combination of bars is used to make an alloy containing 40 kg of iron, 20 kg or nickel and 40 kg of lead.

    Find: The number of small bars needed.

    Each large bar contains 4 kg of iron, 1 kg of nickel and 3 kg of lead.

    Each medium bar contains 2 kg of iron, 1 kg of nickel and 3 kg of lead.

    Each small bar contains 1 kg or iron, 1 kg of nickel and 1 kg of lead.

    Let x = the number of large bars used.

    Let y = the number of medium bars used.

    Let z = the number of small bars used.

    $8x + 6y + 3z = 40 + 20 + 40$

    $8x + 6y + 3z = 100$

    From the given ratios, we can write the following equations for the iron, nickel and lead content of the final alloy:

    $4x + 2y + z = 40$

    $x + y + z = 20$

    $3x + 3y + z = 40$

    Subtracting the equations $4x + 2y + 3z = 40$ and $4x + 2y + z = 40$ yields

    $x - y = 0$

    $x = y$

    Substituting into the equation $x + y + z = 20$ gives us

    $2x + z = 20$

    $z = 20 - 2x$

With y and z in terms of x we can substitute into the original equation to get

$8x + 6x + 3(20 - 2x) = 100$

$14x + 60 - 6x = 100$

$8x = 40$

$x = 5$

The number of small bars needed is

$20 - 2(5) = 20 - 10 = 10$ **Ans.**

23. Given: $x + \frac{1}{x} = 3$

Find: $x^4 + \frac{1}{x^4}$

$$\left(x + \frac{1}{x}\right)^2 = 3^2$$

$$x^2 + 2 + \frac{1}{x^2} = 9$$

$$x^2 + \frac{1}{x^2} = 7$$

$$\left(x^2 + \frac{1}{x^2}\right)^2 = 7^2$$

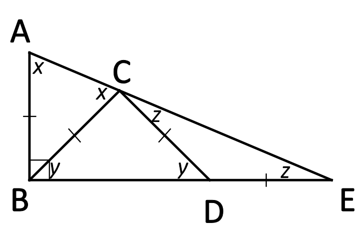
$$x^4 + 2 + \frac{1}{x^4} = 49$$

$x^4 + \frac{1}{x^4} = 47$ **Ans.**

24. Given: Points C and D are on lines AE and BE, respectively.

AB = BC = CD = DE = 1

Find: The measure of $< BAE$



For ΔABC let $x = m\angle BAC = m\angle BCA$.

For ΔBCD let $y = m\angle CBD = m\angle CDB$.

For ΔCDE let $z = m\angle DCE = m\angle DEC$. Then

$m\angle ABC = 180 - 2x = 90 - y$

$y = 2x - 90$

$m\angle BCD = 180 - 2y = 180 - (x + z)$

$2y = x + z$

$m\angle CDE = 180 - 2z = 180 - y$

$2z = y$

For ΔABE $m\angle BAE = x$ and $m\angle AEB = z$ so

$x = 90 - z$

$x + z = 90$

Substituting $2y = x + z$, we get

$2y = 90$

$y = 45$

Now, substituting into $y = 2x - 90$, we get

$2x - 90 = 45$

$2x = 135$

$x = 67.5$ **Ans.**

25. Given: Small drinks cost \$1.20, mediums cost \$1.30 and larges cost \$1.80. Ten people each order one drink costing \$14.90.

Find: The number of large drinks ordered.

Let x = the number of small drinks.

Let y = the number of medium drinks.

Let z = the number of large drinks.

From the information provided, we have

$1.2x + 1.3y + 1.8z = 14.9$

$x + y + z = 10$

Unfortunately, we have more unknowns than we have equations, so we aren't able to use any of the standard techniques we typically employ to solve a system of equations. Let's try another approach.

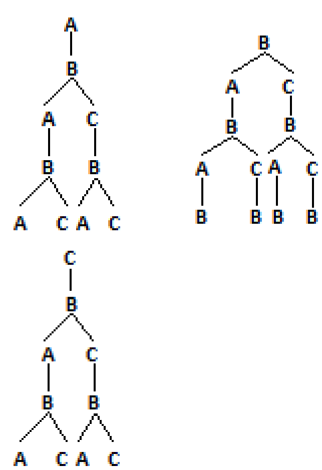
The maximum number of large drinks that could be ordered is 8 since 1.8 × 8 = 14.4. That would leave \$0.50 which isn't enough for another drink of any size.

With seven large drinks, we have 1.8 × 7 = 12.6. That leaves 14.90 – 12.60 = \$2.30. But we can't divide that to buy an integral number of small and/or medium drinks.

With six large drinks, we have 1.8 × 6 = 10.8. That leaves 14.90 – 10.80 = \$4.10. And that still can't be divided to purchase an integral number of small and/or medium drinks.

With five large drinks, we have 1.8 × 5 = 9. That leaves 14.90 – 9.00 = \$5.40. Again, that can't be divided to purchase an integral number of small and/or medium drinks. With four large drinks, we have 1.8 × 4 = 7.2. That leaves 14.90 – 7.20 = \$7.70. If one small drink is ordered, that costs 1.20. That leaves 7.70 – 1.20 = \$6.50 which is enough to buy 6.5 ÷ 1.3 = 5 medium drinks. Thus, the number of large drinks ordered was 4 **<u>Ans.</u>**

26. Given: 25 cells in a 5x5 grid. Each cell has a 0, 1 or 2, so that the values differ by 1.
Find: number of grids.
Let's take a look at the possibilities for the first row.
Let's suppose that the first row starts with 0. We can then have
0 1 0 1 0
0 1 0 1 2
0 1 2 1 0
0 1 2 1 2
Call this Group A.
Now suppose that the first row starts with 1. We can then have
1 0 1 0 1
1 0 1 2 1
1 2 1 0 1
1 2 1 2 1
Call this Group B.
Now suppose that the first row starts with 2. We can then have
2 1 0 1 0
2 1 0 1 2
2 1 2 1 0
2 1 2 1 2
Call this Group C.
Now take one of the items from Group A.
Say, 01010.
What would the second row be?
10101 or
10121 or
12101 or
12121
That's Group B. Any of the items in Group A chosen as the first row will require an item from Group B as the second row.
Now take an item from Group B.
Say, 10101.
What would the second row be?
01010 or
01012 or
01210 or
01212 or
21212 or
21210 or
21010 or
21012
That's Group A and Group C. Any of the items in Group B chosen as the first row will require an item from Group A **or** Group C as the second row.
Now take an item from Group C.
Say, 21212.
What would the second row be?
12121
12101
10121
10101
That's Group A. Any of the items in Group C chosen as the first row will require an item from Group B as the second row. That gives us the following "state" transitions.



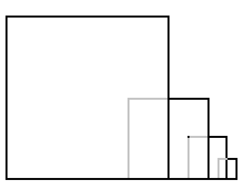
With the first image (Group A is at the top), there are 4 possibilities for the first row, 4 possibilities for the second row (Group B), 8 possibilities for the third row (Group A and C), 4 possibilities for the fourth row (Group B) and 8 possibilities for the fifth row (Group A and C). That's a total of $4 \times 4 \times 8 \times 4 \times 8 = 2^{12} = 4096$ combinations.

With the second image (Group B is at the top), there are 4 possibilities for the first row, 8 possibilities for the second row, 4 possibilities for the third row, 8 possibilities for the fourth row and 4 possibilities for the fifth row. That's another 4096 combinations.

Finally, with the third image, it's similar to the first image for another 4096 possibilities. So 4096 × 3 = 12,288 **Ans.**

27. Given: a square with a side of length 1, another with a side of length $\frac{1}{2}$ bisecting the first square, another with a side of length $\frac{1}{4}$ bisecting the second square, and so on.

Find: The area of the figure.



As can be seen from the diagram, half the area of the second square is added to the entire figure, as is half the area of the third square etc.

Half the area of the second square is

$\frac{1}{2} \times \left(\frac{1}{2} \times \frac{1}{2}\right) = \frac{1}{8}$

Half the area of the third square is

$\frac{1}{2} \times \left(\frac{1}{4} \times \frac{1}{4}\right) = \frac{1}{32}$

Half the area of the fourth square is

$\frac{1}{2} \times \left(\frac{1}{8} \times \frac{1}{8}\right) = \frac{1}{128}$

The area of the entire figure is the sum of the areas of the original square and the remaining half squares. So the sum, let's call it S, is $S = 1 + \frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \cdots$

It follows, then, that $\frac{S}{4} = \frac{1}{4} + \frac{1}{128} + \frac{1}{512} + \cdots$

Subtracting, these two sums, we get

$S - \frac{S}{4} = 1 + \frac{1}{8} - \frac{1}{4}$

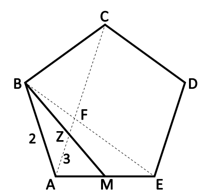
$\frac{3}{4}S = \frac{7}{8}$

$S = \frac{7}{8} \times \frac{4}{3}$

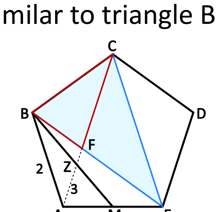
$S = \frac{7}{6}$ **Ans.**

28. Given: Regular pentagon ABCDE with point M as the midpoint of side AE. AC and BM intersect at Z. ZA = 3.

Find: AB

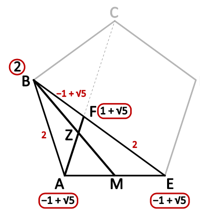


First, consider pentagon ABCDE with diagonal BE, as shown. Let F be the intersection of diagonals AC and BE, each of length *d*. For now, let's assume the pentagon has sides of length 2, and we'll scale this once we determine the actual measurements and ratios. Since AC || DE and BE || CD, it follows that CDEF is a rhombus with CF = DE = EF = DC = 2. Additionally, AF = BF = *d* – 2. Notice, too, that AF = AZ + FZ, so AZ = AF – FZ. We will need this later. If we now draw diagonal CE, as shown, we create isosceles triangle BCE, which is similar to triangle BFC.



We can write the following proportion: *d*/2 = 2/(*d* – 2). Cross-multiplying, we get

$d(d-2) = 4$ and $d^2 - 2d - 4 = 0$. Using the quadratic formula, we see that the solution to this equation is $d = 1 + \sqrt{5}$, so $d - 2 = -1 + \sqrt{5}$. Now, let's apply principles of mass point geometry to triangle BAE with median BM and cevian AF, as shown.



So far, we have AB = FE = 2 and BF = $-1 + \sqrt{5}$. To balance side BE on F, let's assign a mass of 2 at B and a mass of $-1 + \sqrt{5}$ at E. Doing so balances side BE because on one side of F, we have a mass × distance = $2 \times (-1 + \sqrt{5})$ and on the other side of F, mass × distance = $(-1 + \sqrt{5}) \times 2$. This also gives us a mass of $2 + (-1 + \sqrt{5}) = 1 + \sqrt{5}$ at F. Since we've already assigned a mass of $-1 + \sqrt{5}$ to E and M is the midpoint of side AE, we also need to assign a mass of $-1 + \sqrt{5}$ to A. Given the masses at A and F, if cevian AF is balanced on Z, we must have the following ratio AZ/FZ = $(1 + \sqrt{5})/(-1 + \sqrt{5})$.

Cross-multiplying yields

$$FZ(1+\sqrt{5}) = AZ(-1+\sqrt{5})$$

$$FZ = AZ\left(\frac{-1+\sqrt{5}}{1+\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{1-\sqrt{5}}\right)$$

$$FZ = AZ\left(\frac{-6+2\sqrt{5}}{-4}\right)$$

$$FZ = AZ\left(\frac{3-\sqrt{5}}{2}\right)$$

Recall, we previously determined that AZ = AF – FZ and AF = $-1 + \sqrt{5}$. Putting this all together gives us

$$AZ = -1+\sqrt{5} - AZ\left(\frac{3-\sqrt{5}}{2}\right)$$

$$AZ + AZ\left(\frac{3-\sqrt{5}}{2}\right) = -1+\sqrt{5}$$

$$2AZ + AZ\left(3-\sqrt{5}\right) = -2+2\sqrt{5}$$

$$AZ\left(2+3-\sqrt{5}\right) = -2+2\sqrt{5}$$

$$AZ\left(5-\sqrt{5}\right) = -2+2\sqrt{5}$$

$$AZ = \frac{-2+2\sqrt{5}}{5-\sqrt{5}}\cdot\frac{5+\sqrt{5}}{5+\sqrt{5}}$$

$$AZ = \frac{2\sqrt{5}}{5}$$

So, the ratio AZ/AB = $(2\sqrt{5}/5)/2 = \sqrt{5}/5$. Since we are told that AZ = 3, we have the following the proportion $3/AB = \sqrt{5}/5$. Cross-multiplying gives us $\sqrt{5} \times AB = 15$, so AB = $15/\sqrt{5}$. Rationalizing the denominator, we see that AB = $(15/\sqrt{5})(\sqrt{5}/\sqrt{5}) = 15\sqrt{5}/5 = 3\sqrt{5}$ **Ans.**

29. Given: 50 tickets numbered with consecutive integers. Two are drawn.
Find: The probability that the absolute difference between the two numbers is 10 or less.
Suppose one of the tickets chosen contains the number 1. Then a ticket numbered 2 through 11 yields a difference of 10 or less. That's 10 ways to get a difference of 10 or less if one of the tickets is numbered 1. For each of the tickets numbered 1 through 40, if one of these 40 tickets is chosen, then there are 10 ways to select a second numbered ticket such that the numbers differ by 10 or less. That's 40 × 10 = 400 possible ways to select two number tickets in this manner.
Then if the tickets numbered 41 through 50 are chosen, there are 9, 8, 7, 6, 5, 4, 3, 2, 1 and 0 ways, respectively to select a second numbered ticket such that the numbers differ by 10 or less. That's another 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 45 possible ways to select two numbered tickets in the manner described. That's a total of 400 + 45 = 445

possibilities out of the ${}_{50}C_2 = \frac{50!}{(48!)(2!)} = 1225$ possible ways to select two of the numbered tickets.

That's a probability of $\frac{445}{1225} = \frac{89}{245}$ **Ans.**

30. Find: The number of ways to arrange the digits 1 through 9 in a 3 × 3 grid such that the numbers are increasing from left to right in each row and increasing from top to bottom in each column.

Some things we know:

- The first item in the first row must be 1 and the last item in the third row must be 9.
- Whatever works horizontally will work vertically. I.e.,

1 2 3   1 4 7
4 5 6   2 5 8
7 8 9   3 6 9

Let's start with 1 2 __ for the first row.

1 2 3   1 2 3   1 2 3   1 2 3   1 2 3
4 5 6   4 5 7   4 5 8   4 6 7   4 6 8
7 8 9   6 8 9   6 7 9   5 8 9   5 7 9

That's 5.

1 2 4   1 2 4   1 2 4   1 2 4   1 2 4
3 5 6   3 5 7   3 5 8   3 6 7   3 6 8
7 8 9   6 8 9   6 7 9   5 8 9   5 7 9

That's 5.

1 2 5   1 2 5   1 2 5   1 2 5   1 2 5
3 4 6   3 4 7   3 4 8   3 6 7   3 6 8
7 8 9   6 8 9   6 7 9   4 8 9   4 7 9

That's 5.

1 2 6   1 2 6   1 2 6   1 2 6
3 4 7   3 4 8   3 5 7   3 5 8
5 8 9   5 7 9   4 8 9   4 7 9

That's 4.

1 2 7   1 2 7
3 4 8   3 5 8
5 6 9   4 6 9   That's 2.

That's a total of 21.

Note that in each set of 9, the first column covers the following combinations:

1 3 4, 1 3 5, 1 3 6, 1 3 7, 1 4 5, 1 4 6, 1 4 7

These are the total possibilities for either the first column or row. You can't have 8 as the last value in the first row or column. And 1 5 6 leaves too many values that are smaller and we'll violate the increase requirement.

21 × 2 = 42 **Ans.**

Target Round

1. Given: Julia's age is a 2 digit number. The remainder is 1 when her age is divided by 2, 3, 4, 6 or 8. Julia is 5 times Bart's age.
   Find: Bart's age
   Let x be Julia's age. Then, when $x - 1$ is divided by 2, 3, 4, 6 or 8, the remainder is 0. What we are looking for is the Least Common Multiple which is: $2 \times 3 \times 2 \times 2 = 24$. Therefore, $x = 24 + 1 = 25$. Julia's age is 5 times Bart's age so Bart must be $\frac{25}{5} = 5$ **Ans.**

2. Given: p be the maximum number of points of intersection of n distinct lines. $p:n = 6:1$
   Find: n
   If we begin drawing lines, we see that each new line can adds, at most, a number of points of intersection equal to the number of lines drawn.

No. Lines	Max Pts.
1	0
2	1
3	3
4	6
5	10

The maximum number of points of intersection for n lines follows this pattern which is given by the formula:

$$\frac{n(n-1)}{2}$$

We are looking for the number of lines n for which this ratio is 6:1, so we have

$$\frac{\frac{n(n-1)}{2}}{n} = \frac{6}{1}$$

$$6n = \frac{n(n-1)}{2}$$

$12n = n^2 - n$

$n^2 = 12n + n = 13n$

$n = 13$ **Ans.**

3. Given: p is the greatest prime whose digits are distinct prime numbers
   Find: The units digit of p^2
   The only one-digit prime numbers are 2, 3, 5, and 7. We are looking for a large prime number with digits that consist of at least two and at most four of these. We know the last digit cannot be 2 or 5 because any number that is two or more digits and ends in 2 or 5 will not be prime. This means our last digit must be 3 or 7. The squares of both 3 and 7 have a units digit of 9 **Ans.**

4. Given: $\frac{a}{4-a} = \frac{b}{5-b} = \frac{c}{7-c} = 3$
   Find: $a + b + c$

$a = 3(4 - a)$	$b = 3(5 - b)$	$c = 3(7 - c)$
$a = 12 - 3a$	$b = 15 - 3b$	$c = 21 - 3c$
$4a = 12$	$4b = 15$	$4c = 21$
$a = 3$	$b = \frac{15}{4}$	$c = \frac{21}{4}$

$a + b + c = 3 + \frac{15}{4} + \frac{21}{4}$

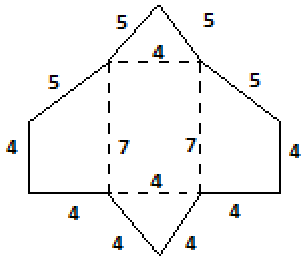
$= 3 + \frac{36}{4}$

$= 3 + 9 = 12$ **Ans.**

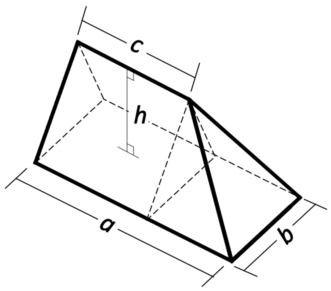
5. Given: The 1-month anniversary of a building was in February, 2000. The 12-month anniversary was in January, 2001. During the year n, the building will celebrate its n month anniversary.
   Find: n
   $(n - 2000) \times 12 = n$
   $12n - 24{,}000 = n$
   $11n = 24{,}000$
   $n \approx \frac{24000}{11} \approx 2181.18$
   In 2181, the building will be 181 years old. January, 2181 is $181 \times 12 = 2172$ months from January, 2000. Nine months later is the 2181th month. 2181 **Ans.**

6. Given: The shape can be folded and taped to form a 3-dimensional polyhedron.

Find: The volume of the polyhedron.



When you make the folds you end up with a wedge, which is polyhedron defined by two triangle and three trapezoidal faces.



We can break this shape into a triangular prism and rectangular pyramid.

The formula for the volume of a wedge is:

$V = \frac{1}{2}bhc$

$a = 7$

$b = 4$

$c = 4$

The height of the polyhedron, h, also is the height of the equilateral triangle at the left end of the polyhedron.

$h = 4 \times \frac{\sqrt{3}}{2} = 2\sqrt{3}$

$V = \frac{1}{2}(4)(2\sqrt{3})(4) = 16\sqrt{3}$

Now we need to find the volume of the pyramid at the end. The formula for volume of a pyramid is:

$V = \frac{1}{3}Bh$

$B = b(a - c) = 4 \times 3 = 12$

$V = \frac{1}{1} \times 12 \times 2\sqrt{3} = 8\sqrt{3}$

Therefore the volume of the entire figure is the sum of the volume of the triangular prism and the pyramid.

$V = 16\sqrt{3} + 8\sqrt{3} = 24\sqrt{3}$ **Ans.**

7. Given: 1-Alice is 15 years younger than twice Catherine's age.

2- Beatrice is 12 years older than half of Alice's age.

3- Catherine is 8 years younger than Beatrice.

4- The sum of the three ages is 100.

But only 3 of the 4 items are true.

Find: Beatrice's age which is an integer.

Let a be Alice's age.

Let b be Beatrice's age.

Let c be Catherine's age.

The four statements can then be written algebraically as follows:

$a = 2c - 15$

$b = \frac{1}{2}a + 12$

$c = b - 8$

$a + b + c = 100$

Let's assume 1, 2 and 4 are true and try to solve the system of equations.

$2c = a + 15$

$c = \frac{a+15}{2}$

$a + \frac{1}{2}a + 12 + \frac{a+15}{2} = 100$

$2a + a + 24 + a + 15 = 200$

$4a + 39 = 200$

Alice's age isn't going to be an integer so it's not items 1, 2 and 4. Let's try items 1, 3, and 4.

$a = 2c - 15$

$c = b - 8$

$a + b + c = 100$

$a = 2c - 15 = 2(b - 8) - 15$

$a = 2b - 31$

$2b - 31 + b + b - 8 = 100$

$4b - 39 = 100$

Again, we're not going to get an integer value. Let's try items 2, 3 and 4.

$b = \frac{1}{2}a + 12$

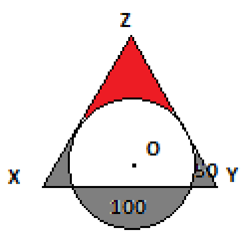
$c = b - 8$

$a + b + c = 100$

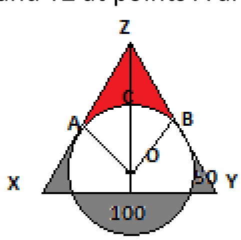
$2b = a + 24$

$a = 2b - 24$
$2b - 24 + b + b - 8 = 100$
$4b - 32 = 100$
$b = \frac{132}{4} = 33$
$c = b - 8 = 33 - 8 = 25$
$a = 2b - 24 = 66 - 24 = 42$
All three ages are integers. $b = 33$ **Ans.**

8. Given: Circle O is tangent to 2 sides of equilateral triangle XYZ. The area of the two shaded regions are 50 and 100 cm as indicated.
Find: The ratio of the area of triangle XYZ to the area of circle O.



The grey shaded area at vertex X is also 50 just as the grey shaded area by vertex Y is (because it's an equilateral triangle).
Draw two lines from the center of the circle that are perpendicular to the tangents XZ and YZ at points A and B.



Let r = the radius of circle O. The area of circle O is πr^2. But this is also the area of ACBYX. (The two grey colored areas of size 50 take the place of the portion of the circle outside of the triangle.)
Quadrilateral AZBO has 360°. The measure of ∠Z is 60°. The measures of both ∠ZAO and ∠ZBO are 90°. Therefore, the measure of ∠AOB is 120°.
Triangles AZO and BZO are similar. The measure of ∠BZO is 30° and the measure of ∠ZOB is 60°. Segment OB, of length r, is opposite the 30° angle so ZB $= r\sqrt{3}$. The same is true for AB. The area of quadrilateral AOBZ is $2 \times \frac{1}{2} \times r \times r\sqrt{3} = r^2\sqrt{3}$. If we subtract the sector of the circle created by ∠AOB we will have found the size of the red-filled area. The sector is just $\frac{1}{3}$ of the area of circle O, or $\frac{1}{3}\pi r^2$. Thus, the red-filled area is $r^2\sqrt{3} - \frac{1}{3}\pi r^2$. The area of triangle XYZ is the red-filled area plus the area of ACBYX.

$$\pi r^2 + r^2\sqrt{3} - \frac{1}{3}\pi r^2 = \frac{2}{3}\pi r^2 + r^2\sqrt{3}$$

The ratio of triangle XYZ to circle O is:

$$\frac{\frac{2}{3}\pi r^2 + r^2\sqrt{3}}{\pi r^2} = \frac{\frac{2}{3}\pi + \sqrt{3}}{\pi} =$$

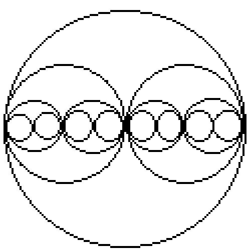
$$\frac{2}{3} + \frac{\sqrt{3}}{\pi} = 0.666666 + 0.55129 \approx$$

1.217995 ≈ 1.22 **Ans.**

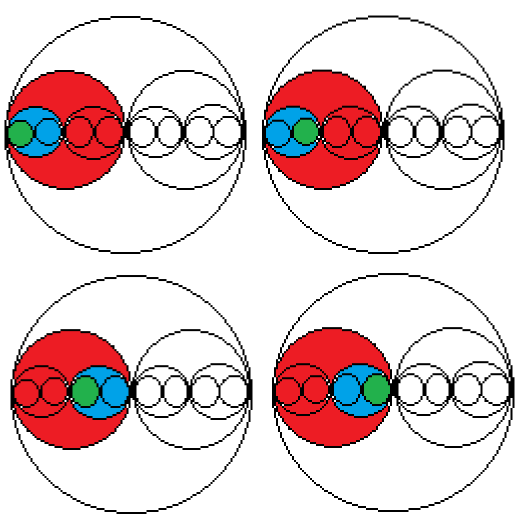
Team Round

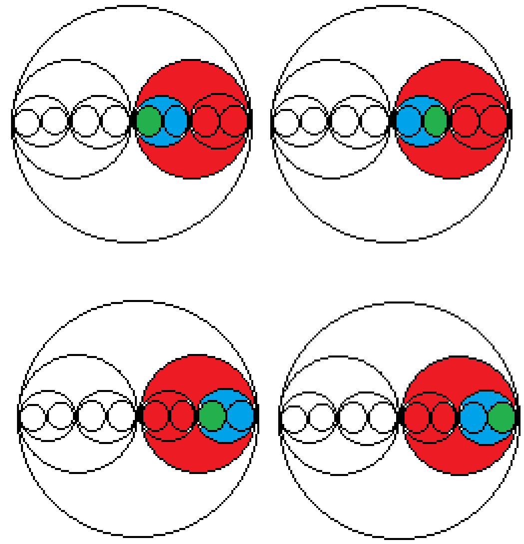
1. Given: The integers 1 through 7 are written in base two.
   Find: The fraction of the digits that are 1.
   The integers 1 through 7 written in base 2 are 1, 10, 11, 100, 101, 110 and 111, respectively. That's a total of 17 digits with 12 of them being 1s. $\frac{12}{17}$ **Ans.**

2. Given: The diagram
   Find: How many sets of 3 distinct circles, A, B and C are there such that circle A encloses circle B and circle B encloses circle C?

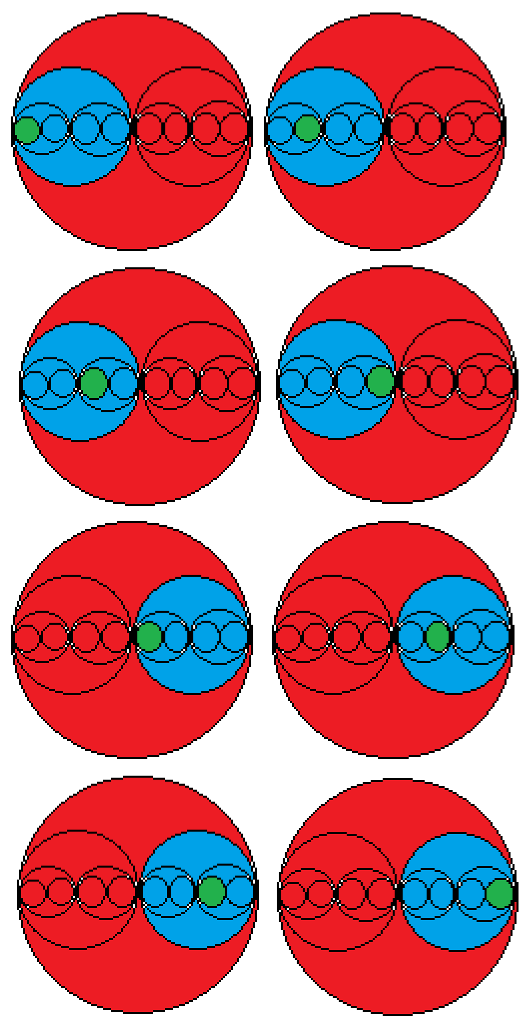


There are 4 sizes of circles – let's call them sizes a, b, c and d where the area of a > the area of b > the area of c > the area of d. There are 8 instances of circles of area b enclosing circles of area c which encloses circles of area d.

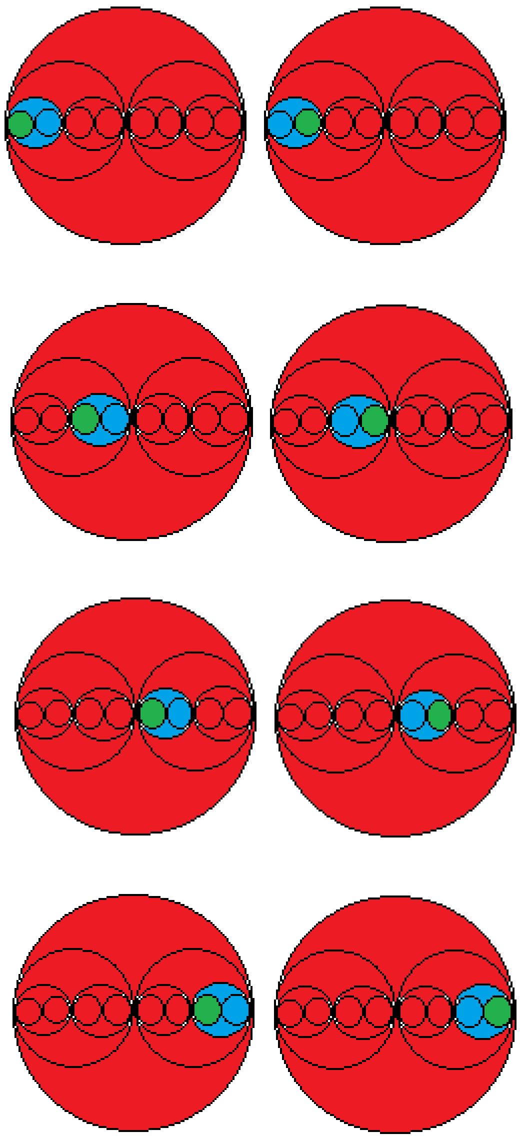




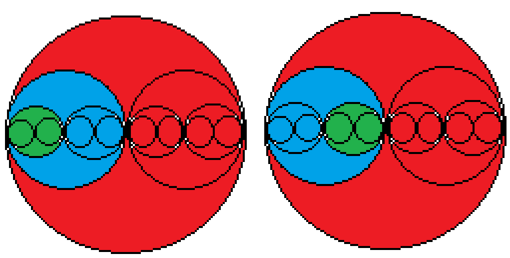
There are also 8 instances of circles of area a > circles of area b > circles of area d.

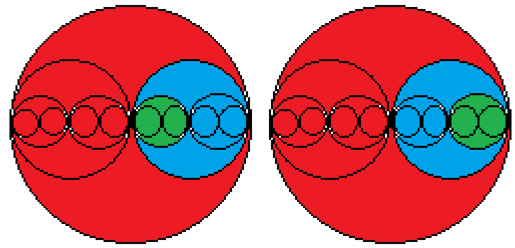


There are also 8 instances of circles of area a > circles of area c > circles of area d.



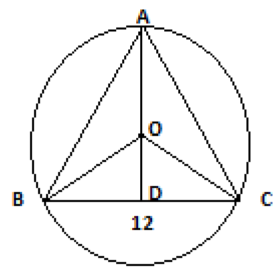
Finally, there are 4 instances of circles of area *a* > circles of area *b* > circles of area *d*.



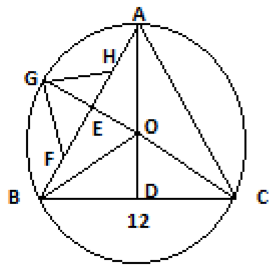


To sum up: 8 + 8 + 8 + 4 = 28 **Ans.**

3. Given: Equilateral triangle ABC is inscribed in a circle. The sides have lengths of 12. Find: the area of the largest equilateral triangle that can be drawn with two vertices on segment AB and the third vertex on minor arc AB.



First, let's find the radius of circle O.
Draw radii from the center of the circle to points A, B, and C. Extend the radius AO to pont D on BC. AD is the height of triangle ABC. BD = DC = 6
AD = $6\sqrt{3}$
BO = *r*
Let *x* = OD.
Triangle BOD is 30-60-90. We know BD=6 so using the known ratios of 30-60-90 triangles we figure out that BO = $4\sqrt{3}$ and OD = $2\sqrt{3}$
The area of the largest triangle that can be drawn with two vertices on segment AB and the third vertex on arc AB that is an equilateral triangle has a height of $2\sqrt{3}$,



We know the height will be this because GO = BO, both the radius of the circle, and EO = OD. Draw triangle FGH. GE is the height.

Let *y* be the length of one of the sides.

We know that ∠F is 60°. That makes

$$\text{GE} = \frac{\sqrt{3}}{2}y = 2\sqrt{3}$$

$$y = \frac{2 \times 2\sqrt{3}}{\sqrt{3}} = 4$$

The area of the triangle is

$\frac{1}{2} \times 4 \times 2\sqrt{3} = 4\sqrt{3}$

$4\sqrt{3}$ **Ans.**

4. Given: A 120-yard by 40-yard rectangular field. In the time it takes Sue to run 120 yards of the perimeter at a rate of 10 minutes/mile, Kara runs the lengths of the other three sides.

   Find: How many minutes it takes Kara to run a mile.

   Sue runs 120 yards. In the same time Kara runs $40 + 120 + 40 = 200$ yards.

   $\frac{200}{120} = \frac{20}{12} = \frac{5}{3}$

   Sue runs at $\frac{5}{3}$ the rate that Kara does.

   So, if Sue takes 10 minutes to run a mile, Kara will take $\frac{3}{5} \times 10 = 6$ minutes to run a mile.

   6 **Ans.**

5. Given: An arithmetic sequence has the first term *a* and common difference *d*. The sum of the first ten tierms is half the sum of the next 10 terms.

   Find: the ratio $\frac{a}{d}$

   The sum of the first 10 terms is:

   $a + (a + d) + (a + 2d) + (a + 3d) + (a + 4d) + (a + 5d) + (a + 6d) + (a + 7d) + (a + 8d) + (a + 9d) = 10a + 45d$

   The sum of the second 10 terms is:

   $(a + 10d) + (a + 11d) + (a + 12d) + (a + 13d) + (a + 14d) + (a + 15d) + (a + 16d) + (a + 17d) + (a + 18d) + (a + 19d) = 10a + 145d$

   $10a + 45d = \frac{1}{2}(10a + 145d)$

   $20a + 90d = 10a + 145d$

   $10a = 55d$

   $\frac{a}{d} = \frac{55}{10} = \frac{11}{2}$ **Ans.**

6. Given: The median of all the change is 50₵. The mean of the change is 40₵. The mode is 40₵.

   Find: the fewest number of students.

   The median is 50₵. Let's assume we have an odd numberof students so that 50₵ is the exact amount in the possession of the middle student. The mode is 40₵ meaning that we have to have at least 2 students with that amount.

   40 40 50 – For 50 to be the median, we need to more values larger than 50 and they can't be the same value (if the mode is caused by 2 values that are the same).

   40 40 50 51 52

   What's the mean?

   $\frac{40 + 40 + 50 + 51 + 52}{5} = \frac{233}{5} = 46.6$

   We need more values, especially smaller ones to get the average down. Let's add 1 and 53

   $\frac{1 + 40 + 40 + 50 + 51 + 52 + 53}{7} = \frac{287}{7} = 41$

Better, but stil not good enough. Add 2 and 54.

$$\frac{1+2+40+40+50+51+52+53+54}{9}$$
$$=\frac{343}{9}<40$$

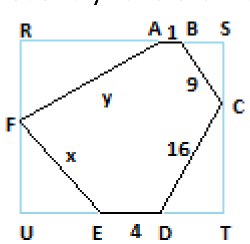
With 9 people we need a total of 360 cents to get the average of 40 and we can do that by changing the 2 to 19, or the 54 to 71 or a number of other values. So 9 people will work. How about 8. In that case we'd need to replace 50 by 49 and 51 and get rid of the 2.

$$\frac{1+40+40+49+51+52+53+54}{8}=$$

$\frac{340}{8}=42.5$ And that's too much.

9 **Ans.**

7. Given: 4 consecutive sides of an equiangular hexagon have lengths of 1, 9, 16 and 4.
Find: the absolute difference in thelengths of the two remaining sides.
Let's "try" and draw this hexagon.



Hexagon ABCDEF has sides 1, 9, 16, 4, x, and y. We're looking for the difference between x and y. In blue, we've also drawn right triangles at each of the points of the hexagon to form rectangle RSTU.
The angles of an equiangular hexagon are each 120°. The measure of ∠FAR, ∠FEU, ∠CDT and ∠CBS are each 60°. The measure of ∠RFA, ∠EFU, ∠DCT and ∠SCB are each 30°. So we have 4 30, 60, 90 triangles.
From this we can determine the lengths of the different segments that make up the rectangle.

$BS=\frac{9}{2}, SC=\frac{9}{2}\sqrt{3}$

$CD=8\sqrt{3}, DT=8$

$UE=\frac{x}{2}, FU=\frac{x}{2}\sqrt{3}$

$FR=\frac{y}{2}\sqrt{3}, RA=\frac{y}{2}$

$RS=UT$

$RS=RA+AB+BS=\frac{y}{2}+1+\frac{9}{2}$

$UT=UE+ED+DT=\frac{x}{2}+4+8$

$\frac{y}{2}+1+\frac{9}{2}=\frac{x}{2}+4+8$

$y+2+9=x+8+16$

$y+11=x+24$

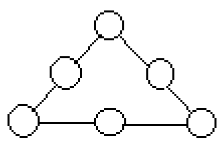
$y-x=24-11=13$

And that's the absolute difference of the lengths of the other two sides. 13 **Ans.**

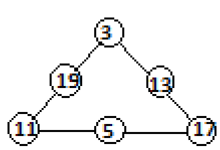
8. Given: A convex sequence is a sequence of integers where each term (but first or last) is no greater than the arithmetic mean of the terms immeidatey before and after it.
Find: How many convex sequences use each number in the set {1, 2, 3, 4, 5, 6, 7, 8} exactly once?
The major thing to note that is that 8 must always be in the first or last position. 7 has to be next to the 8 (which would be second or 7th position or last).
First: the 2 easy sets are:
1— 1, 2, 3, 4, 5, 6, 7, 8
and
2—8, 7, 6, 5, 4, 3, 2, 1
Next we can take 1-6 and surround it by 7 and 8.
3—7, 1, 2, 3, 4, 5, 6, 7, 8
and
4—8, 1, 2, 3, 4, 5, 6, 7
Similarly, we can surround 6-1 with 8 and 7.
5—7, 6, 5, 4, 3, 2, 1, 8
and
6—8, 6, 5, 4, 3, 2, 1, 7

Finally, choosing every other one with half the values decreasing and the others decreasing also gives us a convex sequence.
7—7, 5, 3, 1, 2, 4, 6, 8
and
8 – 8, 6, 4, 2, 1, 3, 5, 7
8 **Ans.**

9. Given: 6 different prime numbers are placed in the circles. The three circles on each side of the triangle have the same sum.
Find: the least possible value of the side sum.



Let's list the primes under 100.
2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97
First of all, 2 can't be one of the numbers. This is because 2 odds plus an even number is an even number but 3 odd numbers sum to an odd number.
So we can start with 3. 3, 5, and 7 differ by 2. But 11, 13, 17 don't differ by 2. 11 and 13 do but 13 and 17 don't. But 3 and 5 do, 11 and 13 do and 17 and 19 do. We can put these together.



3 + 13 + 11 = 11 + 5 + 17 = 3 + 13 + 17 =
33 **Ans.**

10. Given: The sum of two numbers added to the mean of their squares is 64.
Find: the greatest possible value of the product of the two integers.
Let x and y be the numbers.

$$x + y + \frac{x^2 + y^2}{2} = 64$$

Let's write the squares <= 64 and try some substitution.
1 4 9 16 25 36 49 64

$$\frac{36 + 49}{2} + 6 + 7 = \frac{85}{2} + 13 = 42.5 + 13$$
$$= 55.5$$

Too low.

$$\frac{49 + 64}{2} + 7 + 8 = \frac{113}{2} + 15 = 66.5 + 13$$
$$= 79.5$$

Too high.

$$\frac{64 + 36}{2} + 6 + 8 = \frac{100}{2} + 14 =$$
$$50 + 14 = 64$$
$$x = 8; y = 6$$

This is one possible solution.
But notice the wording of the problem: find the "greatest"… It won't work with positive integers which is what we are normally drawn to. Let's try negative integers (because they didn't say we couldn't). And sure enough, try -8, -10.

$$\frac{64 + 100}{2} - 8 - 10 = \frac{164}{2} - 18 =$$
$$82 - 18 = 64$$
$$x = -8; y = -10$$

$-8 \times -10 = 80$ **Ans.**